# Introduction to Elliptic PDEs

# Part II.

for online PDE coffee chat.

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**Problem ③ Continued**

$$\begin{cases} \Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

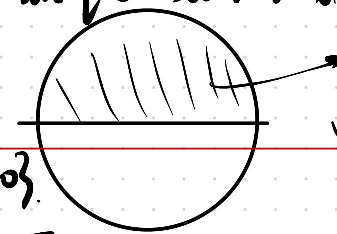
What have we done so far? for $0 < \alpha < 1$.

for $f \in C^{0,\alpha}$ compactly supported, $g \in C(\bar{\Omega})$, $\partial\Omega$ satisfying exterior ball condition, we've established $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ unique solution.

Moreover, using Singular integral approach, we gain interior $C^{2,\alpha}$ estimates.

Today we start with **Global Regularity** and ask whether our unique solution lies in $C^{2,\alpha}(\bar{\Omega})$?

## Step 1 Boundary Estimates on half Balls



this upper half Ball we denote $B_1^+$.

**Lemma** let $u \in C^2(B_1^+) \cap C^0(\overline{B_1^+})$, $u = 0$ on $\{x_n = 0\}$.

Solve $\Delta u = f$ $B_1^+$. with $f \in C^{0,\alpha}(\overline{B_1^+})$ with compact support in $B_1$.

then we want estimate
$$[u]_{C^{2,\alpha}(\overline{B_{1/2}^+})} \le C(n,\alpha)\left( \|u\|_{C^0(\overline{B_1^+})} + \|f\|_{C^{0,\alpha}(\overline{B_1^+})} \right) \quad (*)$$

All bad things happen because there could be "cut" in the force term.

our force $f$.

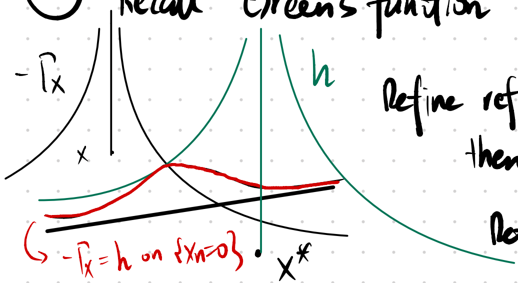
Strategy: to extend force, Newtonian Potential and Harmonic Part.

Morally we still want $u = w + v$, $w = \Gamma * f$, $v$ harmonic function.

**Leverage** $u=0$ $\{x_n=0\}$. design $\tilde{w}$ so $\tilde{w}=0$ on $\{x_n=0\}$. thus $\tilde{v} = u - \tilde{w} = 0$ on $\{x_n=0\}$.

use Green's function — use Schwartz Reflection.

(1) Recall Green's function $G_{\mathbb{R}^n_+}(x,y) := \Gamma_x(y) + h(y)$ where $\begin{cases} \Delta h = 0 & \mathbb{R}^n_+ \\ h = -\Gamma_x & \{x_n=0\} \end{cases}$.

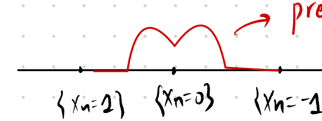



Define reflection point $x^* = (x_1, \cdots, x_{n-1}, -x_n)$ $\forall x \in \mathbb{R}^n_+$.

then $G(x,y) = \Gamma_x(y) - \Gamma_{x^*}(y) = \Gamma(x-y) - \Gamma(x^*-y)$

Define $\tilde{w}(x) := \int_{B_1^+} G(x,y) f(y)\,dy$. so that $\tilde{w} = 0$ $\{x_n=0\}$.

(2) Reflect force evenly $\tilde{f}(x) = \begin{cases} f(x) & x \in B_1^+ \\ f(x', -x_n) & x \in B_1^- \end{cases}$




$\Rightarrow$ $\tilde{w}(x) = \int_{B_1^+} \left(\Gamma(x-y) - \Gamma(x^*-y)\right) f(y)\,dy = \int_{B_1^+} \left(\Gamma(x-y) - \underbrace{\Gamma(x-y^*)}_{\text{use radial symmetry}}\right) f(y)\,dy$

note $\int_{B_1^+} \Gamma(x-y^*) f(y)\,dy = \int_{B_1^-} \Gamma(x-y) \tilde{f}(y)\,dy$ by change of variables.

thus $\tilde{w}(x) = \int_{B_1^+} \Gamma(x-y) f(y)\,dy - \int_{B_1^-} \Gamma(x-y) \tilde{f}(y)\,dy = \boxed{2\int_{B_1^+} \Gamma(x-y) f(y)\,dy - \int_{B_1} \Gamma(x-y) \tilde{f}(y)\,dy}$

the above has two parts! $\tilde{f} \in C_0^{0,\alpha}(B_1)$ so the second part has good estimate

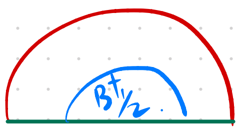
What about first part? let $\overline{w}(x) := \int_{B_1^+} \Gamma(x-y) f(y)\, dy$.

$$\partial_j \overline{w}(x) = \int_{B_1^+} \partial_j \Gamma(x-y) \underbrace{(f(y)-f(x))}_{\text{Hölder regularity ensure well-defined.}} dy + f(x) \boxed{\int_{\partial B_1^+} \partial_i \Gamma(x-y)\, \nu_j \, dS(y)}$$

To see second part,

we use divergence theorem $\int_{B_1^+} \partial_{ij} \Gamma_\varepsilon(x-y)\, dy = \int_{\partial B_1^+} \partial_i \Gamma_\varepsilon(x-y)\, \nu_j \, dS(y)$. $\nu$ outward normal then send $\varepsilon \to 0$

Note $\partial B_1^+ = (\partial B_1 \cap \mathbb{R}^n_+) \cup (B_1 \cap \partial \mathbb{R}^n_+)$




$B_1 \cap \partial\mathbb{R}^n_+$ is what matters.

this doesn't matter since stays $\frac{1}{2}$ distance away.

When $x \in B^+_{1/2}$. (i) But $\nu_j|_{\partial \mathbb{R}^n_+} = 0 \quad \forall j \neq n$. since $\nu = (0, \cdots, 0, -1)$.

$\Rightarrow$ all mixed derivatives contribute $0$ on $\partial\mathbb{R}^n_+$.

and the estimate fully inherits from singular integral in $B_1^+$.

(ii) for pure normal derivative

$$\partial_{nn} \overline{w} = f - \sum_{i=1}^{n-1} \partial_{ii} \overline{w} \quad \text{use equation!}$$

$$\|\partial_{nn} \overline{w}\|_{C^{0,\alpha}(\overline{B}^+_{1/2})} \lesssim \|f\|_{C^{0,\alpha}} + \sum_{i=1}^{n-1} \|\partial_{ii} \overline{w}\|_{C^{0,\alpha}} \lesssim \|f\|_{C^{0,\alpha}}.$$

To conclude for $\tilde{w}$.

$$\|\partial_{ij} \tilde{w}\|_{C^{0,\alpha}(\overline{B}^+_{1/2})} \leq C(n,\alpha) \|f\|_{C^{0,\alpha}(\overline{B}_1^+)}$$

③ for $v = u - \tilde{w}$. $\begin{cases} \Delta v = B_1^+ \\ \boxed{v = 0 \quad B_1 \cap \{x_n = 0\}} \end{cases}$ that $v=0$ $\{x_n=0\}$ is crucial assumption for Schwartz Reflection to preserve harmonicity

Schwartz Reflection (odd reflection) extends $v$ to $\tilde{v}$ defined in $B_1$ s.t. $\begin{cases} \Delta \tilde{v} = 0 & B_1 \\ \tilde{v} = v & B_1^+ . \end{cases}$

$$\tilde{v}(x) := \begin{cases} v(x) & x \in \overline{B_1^+} \\ -v(x', -x_n) & x \in B_1^+ \end{cases}$$

easily check MVP is satisfied on $\{x_n = 0\}$!

Now use gradient estimates for harmonic functions.

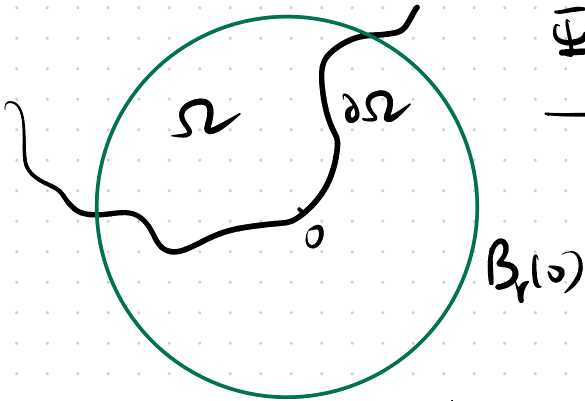
$$\|v\|_{C^{2,\alpha}(\overline{B}^+_{1/2})} \le \|\tilde{v}\|_{C^{2,\alpha}(\overline{B}_{1/2})} \le C(n,\alpha) \|\tilde{v}\|_{C^0(\overline{B}_1)} \le C(n,\alpha) \left( \|u\|_{C^0(\overline{B}_1^+)} + \|f\|_{C^0(\overline{B}_1^+)} \right)$$

Finally, writing $u = \tilde{w} + \tilde{v}$, the desired estimate (*) follows ∎

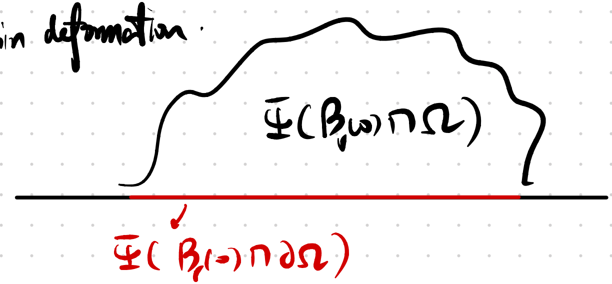
Now we have our Boundary estimates on $\overline{B}^+_{1/2}$.

Can we conclude Global $C^{2,\alpha}$ regularity? NOT NOW!

Assume our $\Omega$ is $C^{2,\alpha}$.




$\xrightarrow{\Phi\ C^{2,\alpha}\text{ domain deformation}}$




Sure. this we can always do.

BUT our equation changes !!!

Say $y = \Phi(x)$ $\Phi$ $C^{2,\alpha}$ local diffeomorphism.

then assume $\Delta u = f$ in $\Omega$. define $\tilde{u}(y) = u(x)$, $\tilde{f}(y) = f(x)$

So $\partial_i u = \partial_k \tilde{u}\, \partial_i \Phi^k$.

$\partial_{ij} u = \partial_{kl}\tilde{u}\, \partial_i \Phi^k \partial_j \Phi^l + \partial_k \tilde{u}\, \partial_{ij} \Phi^k$.

and $\tilde{u}$ solves $\boxed{\partial_i \Phi^k \partial_i \Phi^l \partial_{kl}\tilde{u} + \Delta\Phi^k \partial_k \tilde{u} = \tilde{f}}$

this is not Poisson's equation. and currently we can do nothing about it.

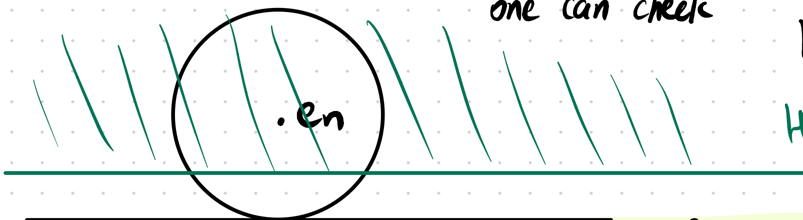
However we can solve on Balls.

## Step 2: Global $C^{2,\alpha}$ Regularity on Balls

There is some special Domain transformation Kelvin Transform that preserves the equation!

If $\Omega = B_1(e_n)$ consider $x \in B_1(e_n) \longmapsto x^* = \frac{x}{|x|^2}$ reflection point.

one can check $\{x_n > \frac{1}{2}\} = \{x^* \mid x \in B_1(e_n)\}$.




Half plane is "Kelvin Transform" of unit Ball.

moreover consider $u^*(x) := |x|^{2-n} u(x^*)$ $\forall x_n > 1/2$. (notice $(x^*)^* = x$)

one may check $\Delta u^* = |x|^{-n-2} f(x^*)$ still solves Poisson's equation in $\{x_n > 1/2\}$

provided $\Delta u = f$ in $B_1(e_n)$

Thus Lemma 1.
Boundary estimate on Half Ball applies. So

Problem (3.1) is solvable

$$\begin{cases} \Delta u = f & B_1 \\ u = g & \partial B_1 \end{cases} \quad f \in C^{0,\alpha}(\bar{B}_1),\ g \in C^{2,\alpha}(\bar{B}_1) \quad \exists!\ u \in C^{2,\alpha}(\bar{B}_1) \quad (**).$$

with $\|u\|_{C^{2,\alpha}(\bar{B}_1)} \leq C(n,\alpha)\left(\|g\|_{C^{2,\alpha}} + \|f\|_{C^{0,\alpha}}\right)$

# Problem 4

$a_{ij}$ uniformly elliptic, i.e., $\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$.

$$\begin{cases} Lu := a_{ij}\partial_{ij}u + b_i\partial_i u + cu = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

We always assume $a_{ij}$ unif. elliptic with elliptic constants $0 < \lambda \le \Lambda < \infty$

**Theorem** If $a_{ij}, b_i, c, f \in C^{0,\alpha}(\bar{\Omega})$, $g \in C^{2,\alpha}(\bar{\Omega})$, $\Omega$ $C^{2,\alpha}$ Domain, and Maximum Principle holds then $\exists!$ $u \in C^{2,\alpha}(\bar{\Omega})$ solution and $\|u\|_{C^{2,\alpha}(\bar{\Omega})} \le C(n, \Omega, \alpha, \lambda, \Lambda)\left(\|f\|_{C^{0,\alpha}(\bar{\Omega})} + \|g\|_{C^{2,\alpha}(\bar{\Omega})}\right)$

Some immediate Remarks

i) $L = \Delta$ with $\Omega$ $C^{2,\alpha}$ is a special case, but solves Problem 3 on general $C^{2,\alpha}$ domains.

ii) to prove the theorem, we need the fact that Problem 3 is solvable on Balls.

The ingredient combines $\begin{cases} \text{Method of Continuity} \\ \text{A priori Estimates.} \end{cases}$

iii) that Maximum Principle needs to hold is essential!

- **Method of Continuity** let $L_0, L_1 : X \to Y$ be bounded linear operators.

$X$: Banach space; $Y$ ↳ normed vector space

consider a family of operators $\mathcal{L}_t := (1-t)L_0 + tL_1 : X \to Y$.

Assume that we know

(i) $\|x\|_X \le C \cdot \|\mathcal{L}_t x\|_Y \quad \forall t \in [0,1], \ \forall x \in X$

(ii) $L_0$ is invertible, i.e., $L_0$ bijective, $L_0^{-1}$ bounded linear.

Then $L_1$ is invertible.

proof What do we mean by $\mathcal{L}_t$ invertible? $\forall y \in Y. \ \exists! \ x \in X$ s.t. $\mathcal{L}_t x = y$.

But $\mathcal{L}_t x = y \iff L_0 x = y + t(L_0 - L_1)x$

$$\iff x = \underbrace{L_0^{-1} y + t \cdot L_0^{-1}(L_0 - L_1)x}_{\text{Define as } Tx}$$

By assumption (ii) $L_0$ is invertible

$\Rightarrow$ it suffices to show fixed point exists for $T$.

Since $X$ is Banach space, we use Contraction Mapping Theorem: it suffices to show $\|T\| < 1$.

$\forall x_1, x_2 \in X. \ \|Tx_1 - Tx_2\|_X \le t \cdot \|L_0^{-1}(L_0 - L_1)(x_1 - x_2)\|_X \overset{\text{(i) estimate on } L_0^{-1}}{\le} C \, t \, \|(L_0 - L_1)(x_1 - x_2)\|_Y$

$$\le C \cdot t \cdot (\|L_0\| + \|L_1\|) \|x_1 - x_2\|_X < \|x_1 - x_2\|_X$$

By taking $t < \dfrac{1}{2C(\|L_0\| + \|L_1\|)} = \delta$. But this $\delta$ is uniform in $x \in X$, $t \in [0,1]$ $\Rightarrow$ push all the way to $t = 1$

# In practice, what are our $X$, $Y$, $L_0$, $L_1$, and estimate (i)?

① up to subtracting $u-g$, we treat $\mathcal{L}g \in C^{0,\alpha}(\bar\Omega)$ as force. Hence WLOG, assume $\boxed{u=0 \ \partial\Omega}$

Define $X := C^{2,\alpha}_{zero}(\bar\Omega) = \{ u \in C^{2,\alpha}(\bar\Omega) \mid u=0 \text{ on } \partial\Omega \}$. $Y := C^{0,\alpha}(\bar\Omega)$

② $L_0 = \Delta$ Laplace operator is our most natural choice.

$L_1 = \mathcal{L}$ non-divergence form uniformly elliptic operator is our target.

**Q** What's missing to apply Method of continuity?

part of our assumption

Reason it's called "A priori"

(i) A priori Estimates. $\|u\|_{C^{2,\alpha}(\bar\Omega)} \le C \|\mathcal{L}_t u\|_{C^{0,\alpha}(\bar\Omega)}$ $\forall t \in [0,1]$, $\forall u \in C^{2,\alpha}_{zero}(\bar\Omega)$

Recall $\mathcal{L}_t = (1-t)\mathcal{L}_0 + t\mathcal{L}_1$. So we're really looking for estimates

for $u$ solution to either $\begin{cases} \Delta u = f & \Omega \\ u=0 & \partial\Omega \end{cases}$ $\begin{cases} \mathcal{L}u = f & \Omega \\ u=0 & \partial\Omega \end{cases}$ that $\|u\|_{C^{2,\alpha}(\bar\Omega)} \le C\left(\|u\|_{C^0(\bar\Omega)} + \|f\|_{C^{0,\alpha}(\bar\Omega)}\right)$

$\le C \cdot \left(\|f\|_{C^{0,\alpha}(\bar\Omega)}\right)$

requires maximum principle.

(ii) Invertibility of $L_0 = \Delta$

we've solved Problem ③ on Balls. But we haven't done so for general Domains.

## A priori Estimates

part of the assumption!

**Theorem** Let $u \in C^{2,\alpha}_{zero}(\bar\Omega)$ be solution to $\mathcal{L}u = a_{ij}\partial_{ij}u + b_i\partial_i u + cu = f$ $\Omega$
with $a_{ij}$ unif elliptic. and $a_{ij}, b_i, c, f \in C^{0,\alpha}(\bar\Omega)$. $\Omega$ $C^{2,\alpha}$.
Then $\exists\ C = C(n, \alpha, \Omega, \text{data}) > 0$ s.t.

$$\|u\|_{C^{2,\alpha}(\bar\Omega)} \le C\left(\|u\|_{C^0(\bar\Omega)} + \|f\|_{C^{0,\alpha}(\bar\Omega)}\right)$$

notice we do <u>not</u> assume for maximum principle.

Let's only prove for the interior version. Method of Freezing Coefficients.

**Lemma** Let $u \in C^2(B_1)$ solve $a_{ij}\partial_{ij}u + b_i\partial_i u + cu = f$ $B_1$. $a_{ij}$ unif elliptic. data $\in C^{0,\alpha}(\bar B_1)$
then $\exists\ \delta = \delta(n, \alpha) > 0$ small, and $C = C(n, \alpha, \lambda) > 0$
s.t. whenever $\|a_{ij} - \delta_{ij}\|_{C^{0,\alpha}}, \|b_i\|_{C^{0,\alpha}}, \|c\|_{C^{0,\alpha}} \le \delta$ $(\ast)$
one has $\|u\|_{C^{2,\alpha}(\bar B_{1/2})} \le C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(\bar B_1)}\right)$

$(\ast)$ is to assume our operator $\mathcal{L}$ is "small perturbation" of Laplace
Hence the estimate should "morally" look the same as that of Laplace's.
Why makes sense to assume? By rescaling! $u_r(x) = u(rx)$ $\forall x \in B_1$.

proof  WLOG assume $a_{ij}(0)=\delta_{ij}$. then

$$\boxed{a_{ij}(0)\partial_{ij}u} = \underbrace{(a_{ij}(0)-a_{ij})\partial_{ij}u - b_i\partial_i u - cu + f}_{\text{}} =: \boxed{F}$$

this takes the form of Poisson's equation!

So estimate for Laplace writes

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}})} \le C\left(\|u\|_{C^0(\overline{B_{3/4}})} + \|F\|_{C^{0,\alpha}(\overline{B_{3/4}})}\right)$$

In fact this step uses assumption $u\in C^{2,\alpha}$

$$\le C\Big(\|u\|_{C^0} + \underbrace{\|a_{ij}-\delta_{ij}\|_{C^{0,\alpha}}}_{\le\delta}\|\partial_{ij}u\|_{C^{0,\alpha}} + \underbrace{\|b\|_{C^{0,\alpha}}}_{\le\delta}\|\partial_i u\|_{C^{0,\alpha}} + \underbrace{\|c\|_{C^{0,\alpha}}}_{\le\delta}\|u\|_{C^{0,\alpha}} + \|f\|_{C^{0,\alpha}}\Big)$$

Recall Interpolation for Hölder Spaces. $\forall\varepsilon>0$, $\exists C(\varepsilon)>0$ s.t.

$$\|u\|_{C^{2,\alpha}(\bar\Omega)} \le C(\varepsilon)\|u\|_{C^0(\bar\Omega)} + \varepsilon[D^2u]_{C^{0,\alpha}(\bar\Omega)}$$

So $$[D^2u]_{C^{0,\alpha}(\overline{B_{1/2}})} \le C\left(\|u\|_{C^0} + \|f\|_{C^{0,\alpha}}\right) + C\delta[D^2u]_{C^{0,\alpha}(\overline{B_{3/4}})} \qquad (\Delta)$$

it is very annoying that the domain on RHS is larger ......
So we cannot simply absorb.

We can use a trick. $$[D^2u]_{C^{0,\alpha}(\overline{B_{3/4}})} \le C(n)\sup_{B_{1/8}(x)\subseteq B_{3/4}}[D^2u]_{C^{0,\alpha}(\overline{B_{1/8}(x)})}$$

shrink the domain on RHS at the cost of "recentering".

We WLOG assume $\|u\|_{C^0(\overline{B_1})} + \|f\|_{C^{0,\alpha}(\overline{B_1})} \le 1$. By dividing by large constant.

$\boxed{\text{Rescaling}}$ $u_r(x) := u(rx)$ $\forall x \in B_1$ we ask how equation $u_r$ solves.

hitting derivatives on $u_r(\frac{x}{r}) = u(x)$ $\forall x \in B_r$ and use $Lu = f$ $B_r$

to obtain that

$$a_{ij}(rx)\partial_{ij}u_r + rb_i(rx)\partial_i u_r + r^2 c(rx)\, u_r = r^2 f(rx) \quad B_1.$$

my coefficients still satisfy $(\bigstar)$ upon choosing $r$ sufficiently small.

now What happens if I bring my $u_r$ into $(\Delta)$ ?

$$r^{2+\alpha}[D^2u]_{C^{0,\alpha}(\overline{B_{r/2}})} \le C\left(\|u\|_{L^\infty(B_{3r/4})} + r^{2+\alpha}\|f\|_{C^{0,\alpha}(\overline{B_{3r/4}})}\right) + \underbrace{C\cdot\delta}_{\mu}\cdot r^{2+\alpha}\cdot \sup_{B_{r/8}(x)\subseteq B_{3r/4}} [D^2u]_{C^{0,\alpha}(\overline{B_{r/8}(x)})}$$

$$\Rightarrow [D^2u]_{C^{0,\alpha}(\overline{B_{r/2}(0)})} \le C\cdot r^{-3} + \mu\cdot[D^2u]_{C^{0,\alpha}(\overline{B_{r/8}(x)})} \text{ for some } B_{r/8}(x) \subseteq B_{3r/4}(0)$$

Refine sequence $x_0 = 0$.

$$\begin{cases} r_k = \frac{1}{2}\cdot\frac{1}{2^{2k}} & \forall k \ge 0 \\ B_{r_{k+1}}(x_{k+1}) \subseteq B_{3r_k/2}(x_k) & \forall k \ge 1 \end{cases}$$

where $[D^2u]_{C^{0,\alpha}(B_{r_{k+1}}(x_{k+1}))}$ achieves sup.

Refine $a_k := [D^2u]_{C^{0,\alpha}(B_{r_k}(x_k))}$.

then we obtain $\boxed{a_k \le M\cdot 2^{6k} + \mu\cdot a_{k+1} \;\; \forall k \ge 0}$

Recall our goal is to show $\exists\ \mu$ small, $C>0$ universal s.t. $a_0 \le C$ uniformly in $u \in C^{2,\alpha}(B_1)$.

Assume for contradiction $a_0 > C$ for $C$ of our choice.

then $a_k > C\,\sigma^k$ for $\sigma > 1$ universal

- Base case satisfied
- assume for $k$

$$a_{k+1} \ge \frac{1}{\mu}\left(a_k - M\cdot 2^{6k}\right)$$

$$\boxed{\ge} \frac{1}{\mu}\left(C\cdot\sigma^k - M\,2^{6k}\right) > C\cdot\sigma^{k+1}$$

inductive hypothesis

want to ensure

pick for example $\mu = 2^{-7}$.

$\sigma \in [2^6, 2^7)$

choose $C > \dfrac{M}{1-\mu\sigma}$ sufficiently large

so that $\lim_{k\to\infty} a_k = \infty$.

But the limiting point $x_\infty = \lim_{k\to\infty} x_k \in B_{1/2}$. Reach Contradiction that $u \in C^{2,\alpha}(B_1)$

up to subsequences.

- Boundary a priori estimates follows same procedure from Boundary $C^{2,\alpha}$ estimate for Poisson's Equation on half Balls.
- Global a priori estimates follow by covering with balls and that the class of uniformly elliptic non-divergence form equations with $C^{0,\alpha}$ coefficients remains invariant under $C^{2,\alpha}$ Domain Deformations.

This concludes the proof of Global A priori Schauder estimates

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OK But now. What about Invertibility of $L_0 = \Delta$ ?

With the help of a priori estimates + method of continuity + Problem 3.1 solvable

we solve Problem 4.1
$$\begin{cases} Lu = a_{ij}\partial_{ij}u + b_i\partial_i u + cu = f & B_1 \\ u = g & \partial B_1 \end{cases}$$

If $C^{0,\alpha}$ coefficients, and maximum principle holds

$$\exists!\ u \in C^{2,\alpha}(\bar{B}_1) \text{ with } \|u\|_{C^{2,\alpha}(\bar{B}_1)} \le C(n,\alpha,\lambda,\Lambda)\left(\|f\|_{C^{0,\alpha}(\bar{B}_1)} + \|g\|_{C^{2,\alpha}(\bar{B}_1)}\right)$$

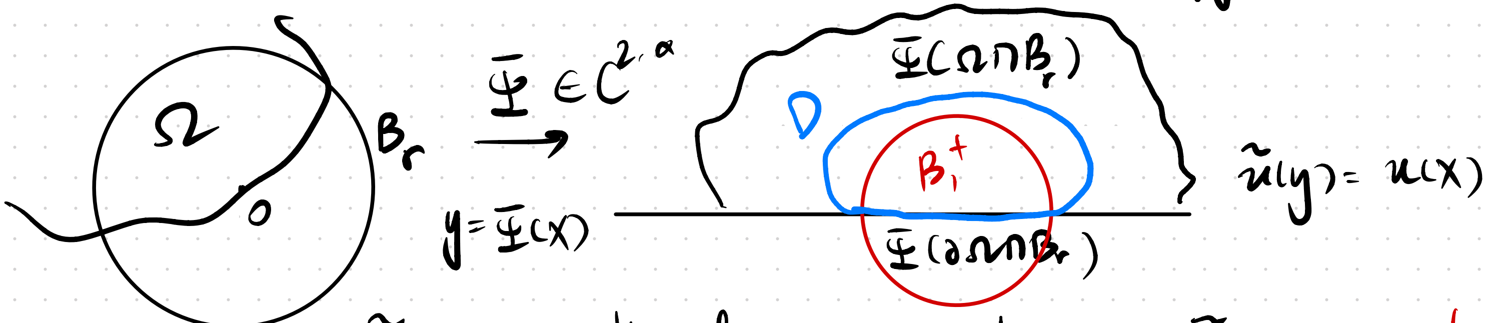
Now. What about general domains $\Omega$ that are $C^{2,\alpha}$?

It suffices to ensure $L_0 = \Delta$ is invertible, i.e., for $f \in C^{0,\alpha}(\bar\Omega)$, $g \in C^{2,\alpha}(\bar\Omega)$

$$\begin{cases} \Delta u = f & \Omega \\ u = g & \partial\Omega, \end{cases} \quad \exists!\ u \in C^{2,\alpha}(\bar\Omega) \text{ s.t. } \|u\|_{C^{2,\alpha}(\bar\Omega)} \le C\left(\|f\|_{C^{0,\alpha}(\bar\Omega)} + \|g\|_{C^{2,\alpha}(\bar\Omega)}\right)$$

In other words, it suffices to solve Problem ③ completely.

What do we have? $u \in C^{2,\alpha}(\Omega) \cap C^0(\bar\Omega)$ need to upgrade to Global $C^{2,\alpha}$.




$\tilde u$ solves $\tilde L \tilde u := \partial_i \Phi^k \partial_i \Phi^l \partial_{kl} \tilde u + \Delta \Phi^k \partial_k \tilde u = \tilde f$ in $B_1^+$

But what is Boundary Data $\tilde u$ on $\partial B_1^+$?

We assume only $\tilde u \in C^2(B_1^+) \cap C^0(\bar B_1^+)$

**Claim 1** One may solve $\begin{cases} \mathcal{L} u = f & D \\ u = g & \partial D \end{cases}$ for $D$ $C^{2,\alpha}$, $D$ diffeomorphic to a Ball.

Problem 4.2

this can be done directly with the help of

a priori estimates + Method of Continuity + $\boxed{L_0 = \Delta \text{ invertible on } D}$

this is resolved due to Problem 4.1 is solvable

**Claim 2** take $D$ as such domain diffeomorphic to Balls that covers $B_1^+$.

By our assumption $\tilde{u}$ is only continuous up to $\partial D$. Call $\varphi = \tilde{u}|_{\partial D}$

Now approximate $\{\varphi_k\} \subseteq C^{2,\alpha}(\bar{D})$ boundary data that is uniformly bounded $C^{2,\alpha}(\bar{D})$.

$$\text{s.t. } \|\varphi_k - \varphi\|_{C^0(\partial D)} \to 0.$$

for each $\varphi_k$ solve $\tilde{u}_k$

$$\begin{cases} \tilde{\mathcal{L}} \tilde{u}_k = \tilde{f} & D \\ \tilde{u}_k = \varphi_k & \partial D \end{cases} \Rightarrow \exists! \ \tilde{u}_k \in C^{2,\alpha}(\bar{D})$$

this is solvable because Problem 4.2 is solvable.

$$\text{with } \|\tilde{u}_k\|_{C^{2,\alpha}(\bar{D})} \le C \left( \|\tilde{f}\|_{C^{0,\alpha}} + \|\varphi_k\|_{C^{2,\alpha}(\bar{D})} \right) \quad (\ast)$$

Now, what about this family of $\{\tilde{u}_k\} \subseteq C^{2,\alpha}(\bar{D})$ ?

- $\begin{cases} \tilde{L}(\tilde{u}_k - \tilde{u}_\ell) = 0 & D \\ \tilde{u}_k - \tilde{u}_\ell = \varphi_k - \varphi_\ell & \partial D \end{cases}$ By Maximum Principle $\|\tilde{u}_k - \tilde{u}_\ell\|_{C^0(\bar{D})} \le \|\varphi_k - \varphi_\ell\|_{C^0(\partial D)} \to 0$

thus $\exists!\ \tilde{u} \in C^0(\bar{D})$ s.t. $\|\tilde{u}_k - \tilde{u}\|_{C^0(\bar{D})} \to 0$.

which of course coincides with our original $\tilde{u}$.

- But one also has the Estimates $(\star)$

$$\|\tilde{u}_k\|_{C^{2,\alpha}(\bar{D})} \le C. \quad \text{uniformly bounded in } k.$$

Now by Ascoli-Arzela. the limit $\tilde{u} \in C^{2,\alpha}(\bar{D})$.

In particular $B_1 \cap \{x_n = 0\} \subseteq \partial D$.

So $\|\tilde{u}\|_{C^{2,\alpha}(\bar{B}^+_{1/2})} \le C(n, \alpha, \Phi, \text{data})\left(\|\tilde{f}\|_{C^{0,\alpha}(\bar{B}^+_1)} + \|\tilde{g}\|_{C^{2,\alpha}(\bar{B}^+_1)}\right)$

With the Boundary estimates one may go to Global $C^{2,\alpha}$ estimates

$\Rightarrow$ Problem 3 is solvable for $u \in C^{2,\alpha}(\bar{\Omega})$

$\Rightarrow$ Problem 4 is solvable for $u \in C^{2,\alpha}(\bar{\Omega})$

using a priori estimate + method of continuity + Problem 3 solvable

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A Remark on importance of Maximum Principle.

consider for example
$$\begin{cases} u'' + u = 0 & (0,\pi) \\ u(0) = 0 \\ u(\pi) = 1 \end{cases}$$

general solution takes the form
$$u(t) = A\sin(t) + B\cos(t)$$

But
$$u(0) = B = 0$$
$$u(\pi) = -B = 1$$

contradiction. $\Rightarrow$ NO solution to this Dirichlet Problem.

$$\boxed{\begin{array}{l}\text{Maximum Principle holds in general when} \\ c \le 0 \text{ or for some } c_0 = c_0(n,\alpha,\Omega) > 0,\ c \le c_0\end{array}}$$