Introduction to Elliptic PDEs

Part I.

for online PDE coffee chat

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<u>Agenda</u> We'll talk about the "Dirichlet Problem" for

① Laplace Equation

② Poisson's Equation

- What is ellipticity?

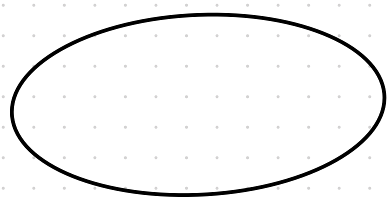
Suppose one is given matrix $A = (a_{ij})$. What does $\{ x \in \mathbb{R}^n \mid x^T A x = 1 \}$ look like in $\mathbb{R}^n$?

since $x^T A x = x^T \left(\frac{1}{2} (A + A^T) \right) x$ it suffices to consider for A symmetric.

Say $n = 2$. $(x, y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = a x^2 + 2 b x y + c y^2 = 1$

if $\det(A) = ac - b^2 > 0$

then

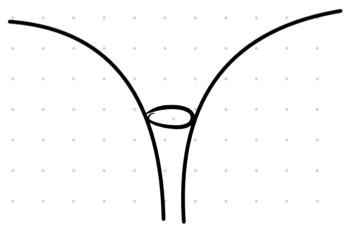


this is your set.

- In general, ellipticity refers to positivity of the leading order coefficient matrix $A = (a_{ij})$
- Simplest Example is $\Delta u = 0$. Laplace Equation

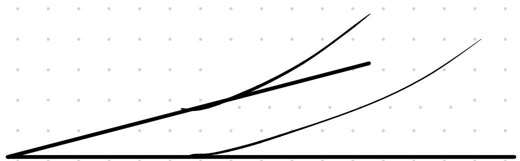
What are some solutions?

1) any linear funtions
2) fundamental solution $\Gamma(x) = \begin{cases} c_n |x|^{2-n} & n \geq 3 \\ c_n \log|x| & n = 2 \end{cases}$.



they're radial global solutions to $\Delta\Gamma = 0$ except at 0.
obtained by solving $\frac{\partial^2}{\partial r^2} u + \frac{n-1}{r}\frac{\partial}{\partial r} u + \frac{1}{r^2}\Delta_{S^{n-1}} u = 0$
and assuming for $u(x) = \Gamma(|x|)$.

3) cone solutions



$f(r,\theta) = r^{\alpha}\cos(\alpha\theta) \quad -\frac{\pi}{2} \leq \alpha\theta \leq \frac{\pi}{2}$.
$= \mathrm{Re}(z^{\alpha})$

Some Key Properties for $u \in C^2(\Omega)$ to $\Delta u = 0$.

① **mean value property** $u(x) = \fint_{B_r(x)} u \, dy = \fint_{\partial B_r(x)} u \, d\mathcal{H}^{n-1}(y)$

too good to wish for for general elliptic PDEs.

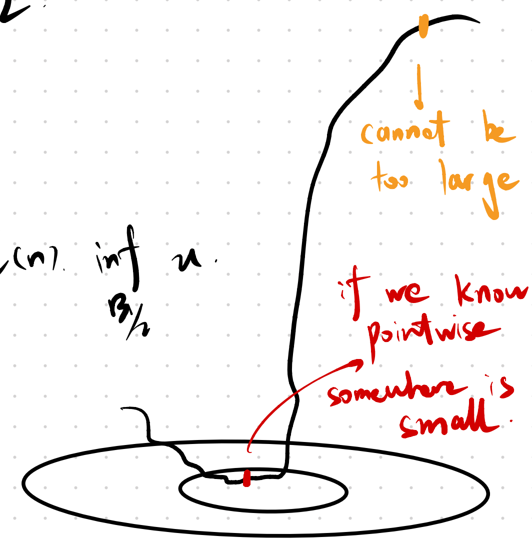
this is "iff" for harmonic functions.

② **Maximum Principle**

$$\begin{cases} \Delta u \geq 0 & \Omega \text{ bounded} \\ u \leq 0 & \partial\Omega \end{cases} \Rightarrow u \leq 0 \quad \Omega.$$

③ **Harnack Inequality**

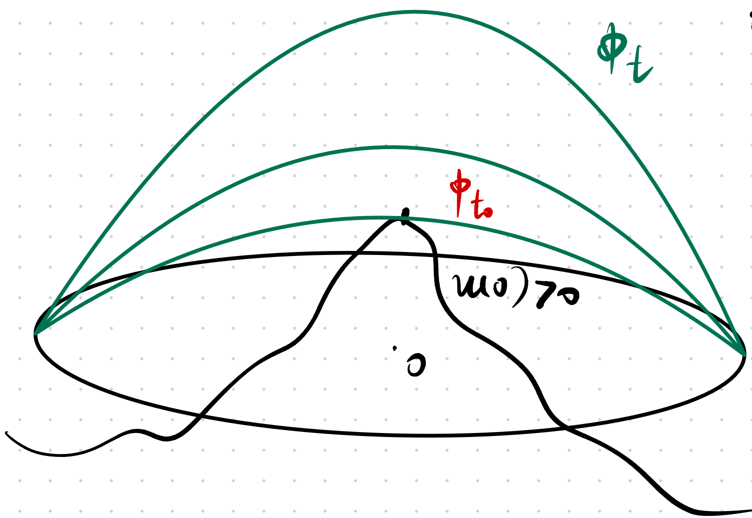
$$\begin{cases} \Delta u = 0 & B_1 \\ u \geq 0 & B_1 \end{cases} \Rightarrow \sup_{B_{1/2}} u \leq C(n) \inf_{B_{1/2}} u.$$



A Quick Proof for Maximum Principle without MVP.

$$\begin{cases} \Delta u \geq 0 & B_1 \\ u \leq 0 & \partial B_1 \end{cases} \Rightarrow u \leq 0 \quad B_1$$

Proof assume at some $0 \in B_1$. $u(0) > 0$. then construct a family of barriers $\phi_t(x) = t(1-|x|^2)$. note $\Delta \phi_t = -2nt < 0$



since $\phi_t \geq u$ on ∂B_1

at some $t_0 > 0$.

$$\begin{cases} \phi_{t_0}(0) = u(0) \\ \phi_{t_0} \geq u \quad B_1 \end{cases}$$

$\Rightarrow$ $\phi_{t_0} - u$ has interior local minimum.

$\Rightarrow$ $\Delta(\phi_{t_0} - u) \geq 0$.

But $0 \leq \Delta u \leq \Delta \phi_{t_0} = -2nt_0 < 0$

Contradiction ! $\square$

- In this talk we study Dirichlet Problem which asks the following
  - Given an equation
  - Given a region where this equation is satisfied
  - Given Boundary Data

$\Rightarrow$

① Existence & Uniqueness
Can I "solve" the equation?
If so, "solve" in what sense? Is the solution unique?

② Interior Regularity
How "smooth" can my solution become in the interior?

③ Boundary Regularity
If my boundary data is sufficiently smooth can my solution approach the boundary data as smooth?

Or even more.

④ Higher Regularity
If my coefficients, domain and boundary data are "smooth enough". Does my solution inherit the smoothness?

⑤ If I cannot solve the problem.
Can I say something about when this happens?

Problem 1 Dirichlet Problem for $\Delta u = 0$ on Balls.

$$\begin{cases} \Delta u = 0 & B_1 \\ u = g & \partial B_1 \end{cases}$$

Method to solve

$$\boxed{\text{Poisson's Integral formula} \quad u(x) = (1-|x|^2) \fint_{\partial B_1} \frac{g(y)}{|x-y|^n} \, d\mathcal{H}^{n-1}(y)}$$

A formal derivation: for u, v smooth, Ω nice.

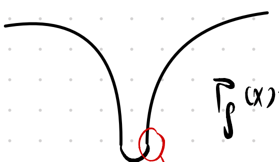
$$\int_\Omega \cancel{\Delta u}^{\,0} \, v = -\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, ds$$

assume $\Delta u = 0$

$$= \int_\Omega u \Delta v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, ds - \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} ds$$

Choose $v = \Gamma_\rho$. let $\rho \to 0$

$$\to \quad 0 = \int_\Omega u \underbrace{\boxed{\Delta \Gamma}}_{= \delta_0} + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \Gamma \, ds - \int_{\partial\Omega} u \frac{\partial \Gamma}{\partial \nu} ds$$



touch in $C^{1,1}$ fashion.

$$\Gamma_\rho(x) := \begin{cases} \Gamma(x) & |x| \geq \rho \\ C^2 \text{ polynomial} & |x| < \rho \end{cases}$$

$\Rightarrow$ with Γ pole centered at $x_0 \in \Omega$.

designed so $\Delta \Gamma_\rho = \frac{1}{|B_\rho|} \chi_{B_\rho}$ this recovers

$$u(x_0) = \int_{\partial\Omega} u \frac{\partial \Gamma_{x_0}}{\partial \nu} ds - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \Gamma_{x_0} \, ds \quad (*)$$

$\Rightarrow \Delta \Gamma_\rho \to \delta_0$ as $\rho \to 0$.

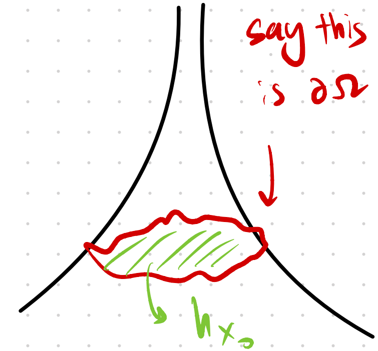
But we want to solve the "Dirichlet problem".

we know $u\big|_{\partial\Omega}$ But not $\frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$ ....

Can I redesign and get rid of this ?

<u>YES!</u> we introduce Green's function $G_{x_0}(y) := \Gamma_{x_0}(y) + h_{x_0}(y)$

where h_{x_0} is harmonic replacement $\begin{cases} \Delta h_{x_0} = 0 & \text{in } \Omega \\ h_{x_0} = -\Gamma_{x_0} & \text{on } \partial\Omega \end{cases}$



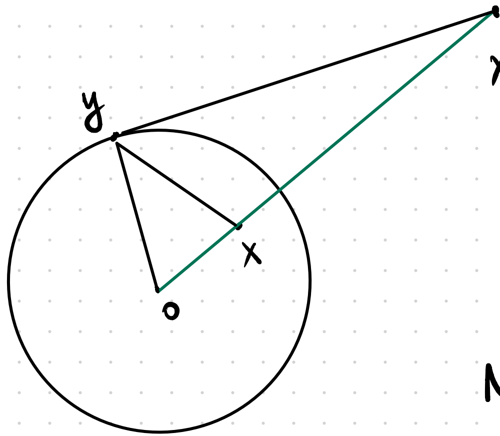
If such "h_{x_0}" exists for Ω

then $(*)$ writes $\boxed{u(x_0) = \int_{\partial\Omega} u(y)\frac{\partial G_{x_0}}{\partial \nu}(y)\, dS_y \quad \text{Green's Representation}}$

Good News! for $\Omega = B_1$. one is able to find h_{x_0} !

How ? By reflecting "the pole" outside B_1.

we ask: How to construct a harmonic function in B_1 that has boundary data $-\Gamma_{x_0}$ on ∂B_1?



$x^* := \frac{x}{|x|^2}$ reflect so that

Recall $\Gamma_{x_0} = C_n |y-x_0|^{2-n}$

$$\Delta y o x \simeq \Delta x^* o y.$$

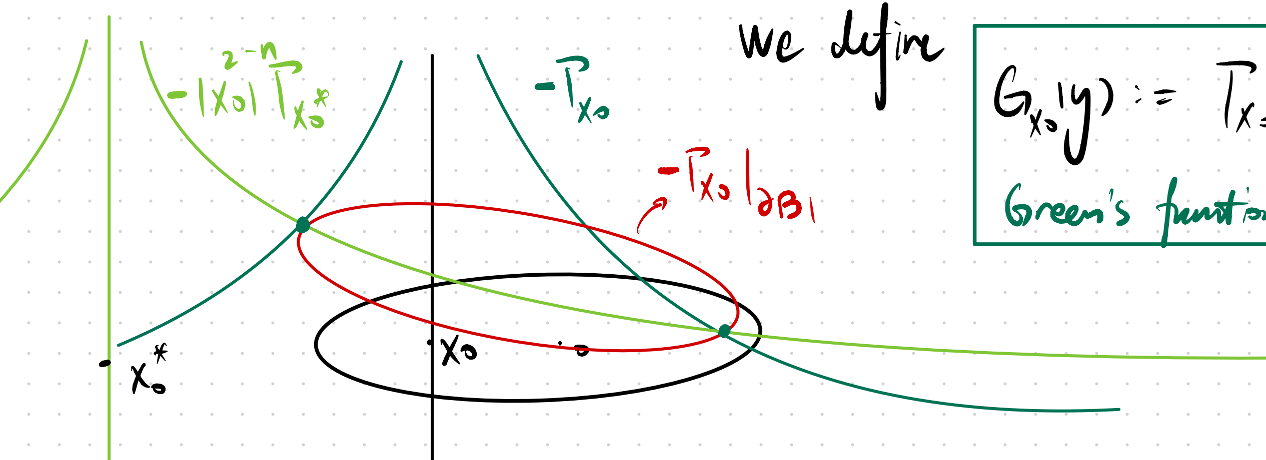
i.e., $\frac{|x^*-y|}{|y|} = \frac{|y-x|}{|x|} \quad \forall y \in \partial B_1$, $1 = |y|$

Now design $h_{x_0} := -|x_0|^{2-n} \Gamma_{x_0^*}(y)$

We define

$$\boxed{G_{x_0}(y) := \Gamma_{x_0}(y) - |x_0|^{2-n} \Gamma_{x_0^*}(y)}$$

Green's function for B_1



look at the Green's Kernel $\nu(y) = \frac{y}{|y|}$ outwards normal on B_1.

$$\frac{\partial G_{x_0}}{\partial \nu_y}(y) = \frac{1}{n\omega_n} \frac{1-|x_0|^2}{|x_0-y|^n}$$

Hence we construct Poisson's Integral formula $u(x_0) = (1-|x_0|^2) \displaystyle\int_{\partial B_1} \frac{g(y)}{|x_0-y|^n} \, d\mathcal{H}^{n-1}(y)$

Interior Regularity Can we differentiate in x_0?

even in general, yes! Can check that $G_{x_0}(y) = G_y(x_0)$.

Green's function is symmetric.

thus for any $y \in \partial B_1$, $\Delta_{x_0} G_y(x_0) = 0$

thus using Dominated Convergence theorem.

$\Delta u(x_0) = 0 \quad \forall x_0 \in B_1$

$\Rightarrow$ By mean value Property we know $u \in C^\infty(B_1)$.

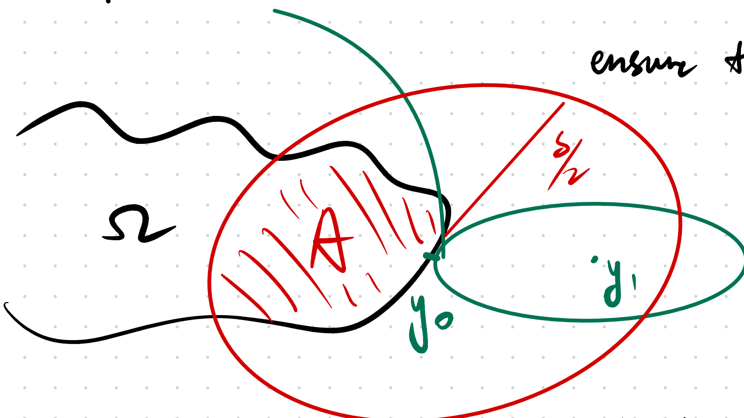
Continuously Approach Boundary Data

Assume $g \in C(\partial\Omega)$

Can I show that $u(x) = \int_{\partial\Omega} g(y) \frac{\partial G}{\partial \nu}x(y)\, d\mathcal{H}^{n-1}(y)$ continuously approach g?

If $\partial\Omega$ has exterior Ball condition, then yes!

ensure that $\boxed{\forall \delta > 0, \quad \sup_{|y-y_0| \geq \delta} \frac{\partial G}{\partial \nu_y}(x) \xrightarrow{x \to y_0} 0}$



$A := B_{\delta/2}(y_0) \cap \Omega$.

One may design Barrier w_{y_1} that satisfies

$$\begin{cases} \Delta w_{y_1} \leq 0 & A \\ w_{y_1}(y_0) = 0 \\ w_{y_1} > 0 & A \end{cases}$$

to trap boundary data $\frac{\partial G}{\partial \nu_y}(x) \leq w_{y_1}(x) \quad \forall x \in \partial A$

$\forall |y - y_0| \geq \delta$

$y_0 \in \partial B_1$.

Now since $\frac{\partial G}{\partial \nu_y}$ is harmonic

and w_{y_1} is super harmonic

$\Rightarrow$ By Maximum Principle $\sup_{|y-y_0| \geq \delta} \frac{\partial G}{\partial \nu_y}(x) \leq w_{y_1}(x) \quad \forall x \in A$

send $x \to y_0$.

Why bother doing the above? use $\int_{\partial\Omega} \frac{\partial G}{\partial \nu}x(y)\, dS(y) = 1$

$$|u(x) - g(x_0)| = \left| \int_{\partial\Omega} (g(x) - g(x_0)) \frac{\partial G}{\partial \nu}x(y)\, dS(y) \right|$$

$$\leq \int_{|y-y_0|\leq\delta} |g(x) - g(x_0)| \left|\frac{\partial G}{\partial \nu}\right| + \int_{|y-y_0|\geq\delta} |g(x) - g(x_0)| \left|\frac{\partial G}{\partial \nu}y(x)\right|$$

$\downarrow$ 0 b/c $g \in C(\partial B_1)$

$\downarrow$ 0 due to uniform convergence outside δ-neighborhood.

$\Rightarrow \longrightarrow 0$ as $X \to X_0$.

Thus we conclude $u \in C^\infty(B_1) \cap C^0(\overline{B}_1)$

Uniqueness Do I have uniqueness?

Yes. By B_1 is bdd domain & maximum principle

Problem ② Dirichlet Problem for $\Delta u = 0$ on Ω with exterior Ball condition

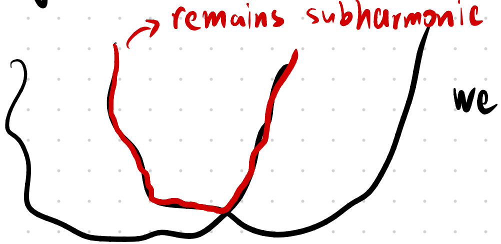
$$\begin{cases} \Delta u = 0 & \Omega \\ u = g & \partial\Omega \end{cases}$$

for example. Ω C^2 has exterior ball condition.

Method to solve

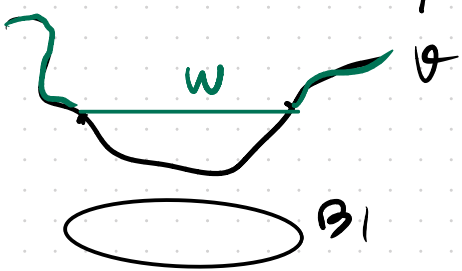
$$\boxed{\text{Perron's Method} \quad u(x) := \sup\left\{ v(x) \;\middle|\; \begin{array}{l} v \le g \text{ on } \partial\Omega \\ v \text{ subharmonic in } \Omega \\ v \in C(\bar{\Omega}) \end{array} \right\}}$$

key important fact: subharmonic functions remain subharmonic upon taking maximum.



we also define harmonic lifting w for any v s.t. $\Delta v \ge 0$.

as $\begin{cases} \Delta w = 0 & B_1 \\ w = v & \partial B_1 \end{cases}$ → this is solvable due to our Problem ①



use MVP one can see $v \le w$ in B_1

thus max $\{w, v\}$ over $\Omega \supseteq B_1$ remains subharmonic function.

Claim $u(x)$ is harmonic in Ω $\Rightarrow$ $u \in C^\infty(\Omega)$ interior regularity

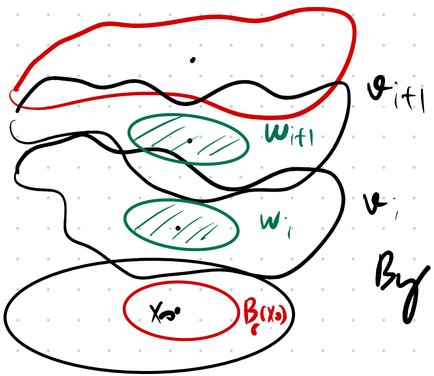
these constants are subharmonic

first of all, the admissible set is nonempty because $m := \inf_{\partial\Omega} g \le v \le M := \sup_{\partial\Omega} g$

Due to definition via "sup", one may take "maximizing sequence" to u.

Pick any $x_0 \in \Omega$ and $B_r(x_0) \subseteq \Omega$. Choose $\{v_i\}$ "admissible" s.t. $\lim_{i\to\infty} v_i(x_0) = u(x_0)$

take $\{w_i\}$ as harmonic lifting of v_i in $B_r(x_0)$



then $\{w_i\}$ remains "admissible"

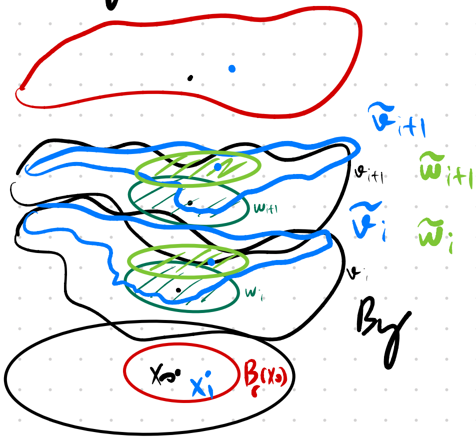
moreover
$$\begin{cases} m \le v_i \le w_i \le u & B_r(x_0) \\ \lim_{i\to\infty} v_i(x_0) = \lim_{i\to\infty} w_i(x_0) = u(x_0) & \\ \Delta w_i = 0 & B_r(x_0) \end{cases}$$

By uniform convergence of harmonic functions preserve "harmonicity" and Arzelà-Arzela (interior gradient estimate) we know $w := \lim_{i\to\infty} w_i$ remains harmonic in $B_r(x_0)$ up to subsequences

But we only know $w(x_0) = u(x_0)$ at this single point.

If we know $w = u$ in $B_r(x_0)$ we're happy ......

To see this, take another point $x_1 \in B_r(x_0)$, $x_1 \neq x_0$.



Repeat the same procedure as above, just this time take "maximizing sequence" $\{\tilde{v}_i\}$ s.t. $\tilde{v}_i \geq w_i \quad \forall i$.

then take $\{\tilde{w}_i\}$ harmonic lifting of $\tilde{v}_i$ in $B_r(x_0)$

By

$$\begin{cases} m \leq v_i \leq w_i \leq \tilde{v}_i \leq \tilde{w}_i \leq u. \\ \lim_{i\to\infty} \tilde{v}_i(x_1) = \lim_{i\to\infty} \tilde{w}_i(x_1) = u(x_1) \\ \lim_{i\to\infty} \tilde{v}_i(x_0) = \lim_{i\to\infty} \tilde{w}_i(x_0) = \tilde{w}(x_0) = w(x_0) = u(x_0) \end{cases}$$

By Arzelà-Ascoli. $\tilde{w} := \lim_{i\to\infty} \tilde{w}_i$ is another harmonic function in $B_r(x_0)$

But $$\begin{cases} \Delta \tilde{w} = \Delta w = 0 \quad B_r(x_0) \\ \tilde{w}(x_0) = w(x_0) \end{cases}$$

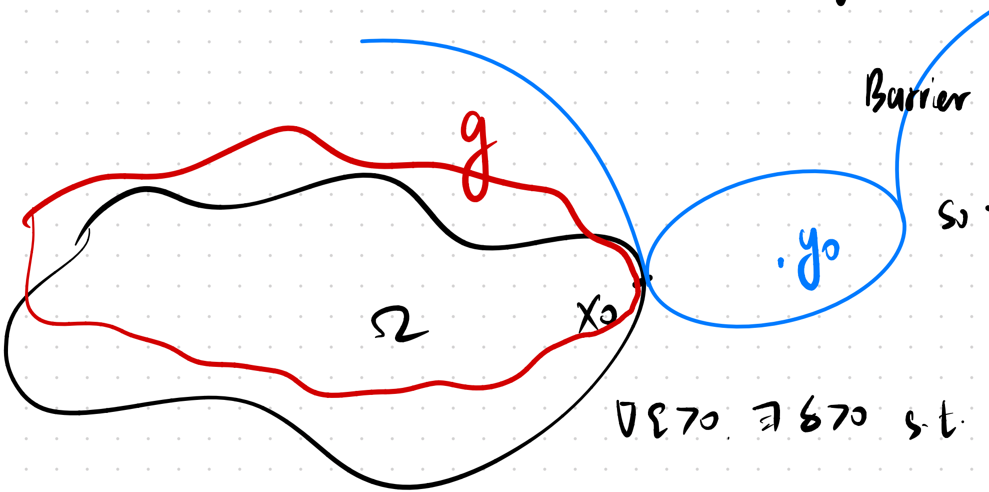
By Strong Maximum Principle we know $\tilde{w} = w$ in $B_r(x_0)$

In particular at x_1. $u(x_1) = \tilde{w}(x_1) = w(x_1)$. But x_1 is arbitrary $\blacksquare$

Continuously Approach Boundary Data

let $g \in C(\partial\Omega)$

Can we show that u continuously approach Boundary Data g?



Barrier $w_{x_0}(y) := \Gamma_{y_0}(y) - \underbrace{\Gamma_{y_0}(x_0 - y_0)}_{\text{constant}}$

so that
$$\begin{cases} \Delta w_{x_0}(x) \le 0 & \Omega \\ w_{x_0}(x_0) = 0 & \\ w_{x_0} > 0 & \Omega \end{cases}$$

$\forall \varepsilon > 0 \;\exists \delta > 0$ s.t. $|g(y) - g(x_0)| < \varepsilon \quad \forall |y - x_0| < \delta$

if on the other hand $|y - x_0| \ge \delta$, $|g(y) - g(x_0)| \le 2 \sup_{\partial\Omega} |g| \le K\, w_{x_0}(y)$

K large enough

$w_{x_0}(y)$ this does not degenerate because $|y - x_0| \ge \delta$

$$\Rightarrow \quad -\varepsilon - K w_{x_0}(y) \le g(y) - g(x_0) \le \varepsilon + K w_{x_0}(y)$$

$$\forall y \in \partial B_1$$

Our goal is to replace the above $g(y)$ with $u(y)$, and for $y \in \Omega$

→ $\underbrace{-\varepsilon - K w_{x_0}(y) + g(x_0)}_{\text{this is subharmonic function below boundary data}} \leq u(y) \quad \forall y \in \Omega$

↳ using construction of u via Perron's method.

→ on the other hand,

$$v(y) \leq g(y) \leq \underbrace{g(x_0) + \varepsilon + K w_{x_0}(y)}_{\text{this is superharmonic}} \quad \forall y \in \partial\Omega$$

$\forall\, v$ admissible subharmonic.

By Maximum Principle

$$v(y) \leq g(x_0) + \varepsilon + K w_{x_0}(y) \quad \forall y \in \Omega$$

But v is arbitrary admissible subharmonic function.

take sup gives u.

→ $|u(y) - g(x_0)| \leq \varepsilon + K w_{x_0}(y)$. first send $y \to x_0$, then $\varepsilon \downarrow 0$.

We conclude $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ ∎

Problem 3

Dirichlet Problem for $\Delta u = f$ Poisson's Equation

$$\begin{cases} \Delta u = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

with Ω exterior Ball condition.

What conditions on f can we make this problem well-posed?

morally. one want to decompose $u = v + w$

where $\Delta w = f \quad \mathbb{R}^n$.

for f compactly supported.

$$\underbrace{\begin{cases} \Delta v = 0 & \Omega \\ v = u - w & \partial\Omega. \end{cases}}$$

this has unique solution from our Problem 2

It suffices to study the behavior of w s.t. $\Delta w = f$.

- Define **Newtonian Potential** $w(x) := \int_{\mathbb{R}^n} f(y)\,\Gamma(x-y)\,dy$

Why is this a reasonable choice?

Recall our derivation

$$\int_\Omega \Delta u\,\Gamma = \int_\Omega u\,\Delta\Gamma_{x_0} + \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\Gamma_{x_0}\,ds - \int_{\partial\Omega} u\,\frac{\partial \Gamma}{\partial \nu}\,ds$$

$$\int_\Omega \underbrace{\Delta u}_{\text{"}f\text{"}}\,G_{x_0} = \int_\Omega u\,\underbrace{\Delta G_{x_0}}_{\text{"}\delta_{x_0}\text{"}} - \int_{\partial\Omega} u\,\frac{\partial G_{x_0}}{\partial \nu}\,ds.$$

Also recall

$$G_{x_0} = \Gamma_{x_0} + h_{x_0} \rightarrow \text{harmonic in } \Omega$$

thus

$$u(x_0) = \underbrace{\int_\Omega f(y)\,\Gamma(x_0-y)\,dy}_{\text{we take Newtonian Potential from here.}} + \text{something harmonic}$$

When is this well-defined? Recall $\Gamma(x) = C_n |x|^{2-n}$

→ when is $w \in C^0$? let $\operatorname{supp}(f) \subseteq B_R$.

$$|w(x)| \leq \int_{B_R} |f| \cdot |\Gamma| \lesssim \|f\|_{L^p} \cdot \int_0^R r^{(2-n)p'} \cdot r^{n-1} \, dr.$$

where $p' = \frac{p}{p-1}$. need $(2-n)\frac{p}{p-1} + n - 1 > -1$

$$\boxed{p > n/2}$$

→ When is $w \in C^1$?

$$|\nabla w(x)| \leq \int_{B_R} |f| |\nabla \Gamma| \lesssim \|f\|_{L^p} \int_0^R r^{(1-n)p'} \cdot r^{n-1} \, dr$$

need $(1-n)\frac{p}{p-1} + n - 1 > -1$

$$\boxed{p > n.}$$

But can we repeat the same procedure for $w \in C^2$? No!

For w to be C^2, we in fact need $f \in C_0^{0,\alpha}$ for $0 < \alpha < 1$.

compact support

<u>formula</u>

$$\boxed{\partial_{ij} w(x) = \frac{\delta_{ij}}{n} f(x) + p.v.(\partial_{ij}\Gamma * f)(x)} \quad (*)$$

where $p.v.(K * f)(x) := \lim_{\varepsilon \to 0} \int_{B_\varepsilon(0)} K(y) f(x-y)\, dy$ for some kernel K.

→ We can develop some theory for $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ that is

- homogeneous of degree $-n$, i.e., $K(x) = |x|^{-n} \underline{g(\tfrac{x}{|x|})}$
- $\int_{\partial B_1} K \, dS_y = 0$

assume $g : \partial B_1 \to \mathbb{R}$ Lipschitz

Indeed $\partial_{ij}\Gamma$ satisfies such requirements for K.

→ <u>Claim: for $f \in C_0^{0,\alpha}$, $Tf(x) := p.v.(K * f)(x)$ is continuous</u>

proof notice $\int_{B_R \setminus B_\varepsilon(0)} K = \int_\varepsilon^R s^{-n} \cdot s^{n-1} \underbrace{\int_{\partial B_1} g \, d\mathcal{H}^{n-1}}_{=0} ds = 0$

thus $\forall \varepsilon > 0$, $\int_{B_R \setminus B_\varepsilon(0)} f(x-y) K(y)\, dy = \int_{B_R \setminus B_\varepsilon(0)} (f(x-y) - f(x)) K(y) \le [f]_{C^{0,\alpha}} \int_\varepsilon^R r^\alpha \cdot r^{-n} \cdot r^{n-1} < \infty$

uniformly in ε

thus $\|Tf\|_\infty \le C$.

moreover $\int_{B_\varepsilon(0)} (f(x-y) - f(x)) K(y) dy \le [f]_{C^{0,\alpha}} \int_0^\varepsilon r^{\alpha - n + n - 1} dr \le [f]_{C^{0,\alpha}} \varepsilon^\alpha \xrightarrow{\varepsilon \to 0} 0$

$\Rightarrow Tf$ is continuous

→ We show the formula (*)

Proof take Γ_ε as our approximation for Γ.

then $\partial_{ij} w^\varepsilon = \partial_{ij}(\Gamma^\varepsilon * f) = (\partial_{ij}\Gamma^\varepsilon) * f = (\partial_{ij}\Gamma^\varepsilon \chi_{B_\varepsilon}) * f + (\partial_{ij}\Gamma^\varepsilon_{B_\varepsilon^c}) * f$

$$= \left(\frac{\delta_{ij}}{n|B_\varepsilon|}\chi_{B_\varepsilon}\right) * f + (\partial_{ij}\Gamma^\varepsilon_{B_\varepsilon^c}) * f$$

$$\xrightarrow{\varepsilon\to 0} \frac{\delta_{ij}}{n} f(x) + \text{p.v.}(\partial_{ij}\Gamma * f)(x) \quad \square$$

We've demonstrated that

for $f \in C_0^{0,\alpha} \Rightarrow w \in C^2$

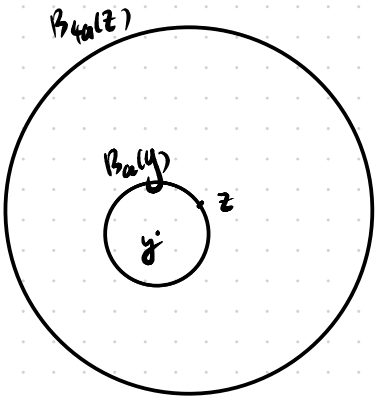
therefore our Problem 3 admits $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ solution

BUT since $f \in C^{0,\alpha}$, why can't $u \in C^{2,\alpha}(\Omega)$???

let's check this!

→ Claim: for $f \in C_0^{0,a}$, in fact $Tf(x) \in C^{0,\alpha}$ $\Rightarrow$ $u \in C^{2,\alpha}(\Omega) \cap C^{0}(\bar{\Omega})$

Proof take arbitrary $y \neq z$, denote $a = |y-z|$



then $$Tf(z) = \int_{\mathbb{R}^n} K(z-x)\big(f(x) - f(z)\big)\,dx$$
$$= \underbrace{\int_{B_{4a}(z)} \cdots}_{o(a^{\alpha})} + \int_{B_{4a}^c(z)} K(z-x)\big(f(x) - f(z)\big)\,dx$$

$$Tf(y) = \int_{\mathbb{R}^n} K(y-x)\big(f(x) - f(y)\big)\,dx$$
$$= \underbrace{\int_{B_a(y)} \cdots}_{o(a^{\alpha})} + \underbrace{\int_{B_{4a}(z)\setminus B_a(y)} K(y-x)\big(f(x) - f(z)\big)\,dx}_{O(a^{\alpha}) \text{ using the ball } B_{4a}(z)} + \int_{B_{4a}^c(z)} K(y-x)\big(f(x) - f(z)\big)\,dx$$

free to switch since away from y

Now subtract both.

$$|Tf(y) - Tf(z)| \le \int_{B_{4a}^c(z)} |K(z-x) - K(y-x)|\,|f(x) - f(z)|\,dx$$
$$\lesssim [f]_{C^{0,\alpha}} \cdot a \cdot \int_{4a}^{\infty} r^{-(n+1)} \cdot r^{\alpha} \cdot r^{n-1}\,dr = o(a^{\alpha}).$$

$\Rightarrow Tf \in C^{0,\alpha}$ $\square$