Some studies on the regularity of solutions to degenerate or singular equations with gradient terms

Jiangwen Wang

Email:jiangwen_wang@seu.edu.cn

Joint works with João Vitor da Silva (UNICAMP), Feida Jiang (SEU) and Yunwen Yin (SEU)

PDE Coffee Chat Organized by Dr.Alvis Zahl(Rutgers University, USA)

Online, April 4, 2026

Outline

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   - Fully nonlinear degenerate parabolic equation

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Fully nonlinear elliptic equations

Theorem (Krylov∼Safonov, 1980)

Let $u \in C(B_1)$ be a viscosity solution to

$$F(D^2u) = 0 \ \text{ in } \ B_1$$

with $F(\cdot)$ uniformly (λ, Λ)-elliptic in the sense that

$$\mathcal{P}^-_{\lambda,\Lambda}(\mathrm{N}) \leq F(\mathrm{M}+\mathrm{N}) - F(\mathrm{M}) \leq \mathcal{P}^+_{\lambda,\Lambda}(\mathrm{N}) \quad \mathrm{M}, \mathrm{N} \in Sym(n).$$

Then $||u||_{C^{1,\alpha_0}(B_{1/2})} \lesssim ||u||_{L^\infty(B_1)}$ for some $\alpha_0 \in (0,1)$.

Fully nonlinear equation has been studied by many leading mathematicians.
▶ Caffarelli, Trudinger, Savin, Silvestre, Teixeira, Figalli, Kriventsov, Xu-Jia Wang, Hongjie Dong, Hui Yu, Lan Tang,$\cdots\cdots$

Degenerate or Singular fully nonlinear elliptic equations

Consider continuous viscosity solutions to

$$|Du|^{\gamma} F(D^2 u) = f \quad \text{in} \quad B_1 \tag{1}$$

with $\gamma \in (-1, \infty)$ and $f \in C(B_1) \cap L^{\infty}(B_1)$.

Theorem (Birindelli∼Demengel, J. Differential Equations., 2010)

As $\gamma \in (-1, 0)$, then $u \in C^{1,\alpha}_{\text{loc}}(B_1)$ for some $\alpha \in (0, 1)$. $\leftrightsquigarrow$ Fixed point methods.

Question

As $0 < \gamma < \infty$, how regular is the viscosity solution to (1) ?

Two heuristic examples

Example

The function $u(x) = |x|^{1+\alpha}$ satisfies

$$|\nabla u|^{\gamma}\Delta u = C(n,\alpha,\gamma)|x|^{(1+\alpha)(1+\gamma)-(2+\gamma)},$$

where $C(n,\alpha,\gamma) = (1+\alpha)^{1+\gamma}(n+\alpha-1)$. Let $\alpha = \frac{1}{1+\gamma}$, then

$$|\nabla u|^{\gamma}\Delta u \equiv \text{constant}.$$

$\Rightarrow$ The solution u of

$$|Du|^{\gamma}F(D^2u) = f \quad \text{in} \quad B_1$$

cannot be more regular than $C^{1,\alpha}$, even for $f \equiv$ constant and $F = \Delta$.

Example

For any given $\theta := (1+\alpha)(1+\gamma) - (2+\gamma) > \gamma$, the function $u(x) = |x_n|^{2+\frac{\theta-\gamma}{1+\gamma}}$ is exactly $C^{2,\frac{\theta-\gamma}{1+\gamma}}$ at the origin (a critical point for u), and satisfies

$$|\nabla u|^\gamma \Delta u = f(x),$$

where

$$f(x) = C(n,\alpha,\gamma)|x_n|^{\gamma+\frac{\gamma(\theta-\gamma)}{1+\gamma}+\frac{\theta-\gamma}{1+\gamma}} = C(n,\alpha,\gamma)|x_n|^\theta \leq C(n,\alpha,\gamma)|x|^\theta.$$

$\Rightarrow$ The solution u can obtain higher regularity at some meaningful points if the Hölder exponent of f is larger than the degenerate rate γ.

Theorem (Imbert∼Silvestre, **Adv. in Math.**, 2013)

As $0 < \gamma < \infty$, then the solution to

$$|Du|^{\gamma} F(D^2 u) = f \quad \text{in} \quad B_1$$

is locally $C^{1,\alpha}$, for some $0 < \alpha \le \frac{1}{1+\gamma}$. $\leftrightsquigarrow$ Geometric tangential methods & Improvement of flatness lemma.

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Theorem (Nascimento, **J. Funct. Anal.**, 2025)

Let $u \in C(B_1)$ be a viscosity solution to

$$|Du|^\gamma F(D^2u) = f \quad \text{in} \quad B_1,$$

where

- f is $C^{0,\theta}$ at 0, $f(0) = 0$ and $\theta > \gamma$;
- 0 is a local extrema for u.

Then $u \in C^{2,\alpha}(0)$ for $\alpha = \frac{\theta-\gamma}{1+\gamma}$. $\leftrightsquigarrow$ Flatness estimate & Scaling techniques & Iterative argument.

Theorem (Araújo∼Ricarte∼Teixeira, Calc. Var. & PDE., 2015)

As $0 < \gamma < \infty$, then the solution to

$$|Du|^\gamma F(D^2u) = f \quad \text{in} \quad B_1$$

is locally $C^{1,\alpha}$, for

$$\alpha = \min\left\{\alpha_0^-, \frac{1}{1+\gamma}\right\}. \rightsquigarrow \text{Approximation method.}$$

In other words, if $\frac{1}{1+\gamma} < \alpha_0$, the solutions are $C^{1,\alpha}(B_{1/2})$ with $\alpha = \frac{1}{1+\gamma}$; if, alternatively, $\alpha_0 \leq \frac{1}{1+\gamma}$, then solutions are $C^{1,\alpha}(B_{1/2})$ for every $\alpha < \alpha_0$.

Optimal regularity

Theorem (Araújo∼Ricarte∼Teixeira, Calc. Var. & PDE., 2015)

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Remark

If F is also concave, then u is locally in $C^{1,\frac{1}{1+\gamma}}$ and this regularity is optimal.

Variable exponent degeneracies/singularities

Theorem (Bronzi∼Pimentel∼Rampasso∼Teixeira, J. Funct. Anal., 2020)

Let $u \in C(B_1)$ be a viscosity solution to

$$|\nabla u|^{\gamma(x)} F(D^2 u) = f \quad \text{in} \quad B_1,$$

where

- $F(\cdot)$ is a (λ, Λ)-uniformly elliptic operator;
- $f \in C(B_1) \cap L^\infty(B_1)$;
- $-1 < \inf_{x \in B_1} \gamma(x) \leq \gamma(\cdot) \in C(B_1)$.

Then $u \in C^{1,\alpha}_{\text{loc}}(B_1)$ for

$$\alpha = \min\left\{\alpha_0^-, \frac{1}{1 + \|\gamma^+\|_\infty + \|\gamma^-\|_\infty}\right\}. \rightsquigarrow \text{Approximation method.}$$

Double Phase type degeneracies

Theorem (De. Filippis, Proc. Roy. Soc. Edinburgh Math., 2021)

Let $u \in C(B_1)$ be a viscosity solution to

$$\big(|Du|^p + a(x)|Du|^q\big)F(D^2u) = f \quad \text{in} \quad B_1,$$

where

- $F(\cdot)$ is a (λ, Λ)-uniformly elliptic operator;
- $f \in C(B_1) \cap L^\infty(B_1)$;
- $0 \le a(\cdot) \in C(B_1)$ and $0 \le p \le q < \infty$.

Then $u \in C^{1,\alpha}_{\text{loc}}(B_1)$ for some $0 < \alpha \le \frac{1}{1+p}$. $\leftrightsquigarrow$ Geometric tangential methods & Improvement of flatness lemma.

Further investigation

Works on the global $C^{1,\alpha}$ regularity and Sobolev regularity

- Dirichlet boundary value problem
  Birindelli∼Demengel, **ESAIM Control Optim. Calc. Var.**, 2014;
  Araújo∼Sirakov, **J. Math. Pures Appl.**, 2024;
  Baasandorj∼Byun∼Lee, **Math. Z.**, 2024;
  W.∼Jiang, Preprint., 2026.
- Neumman boundary value problem
  Banerjee∼Verm, **Potential Anal.**, 2022;
- Oblique boundary value problem
  Byun∼Kim∼Oh, **Calc. Var. & PDE.**, 2025.
- $W^{2,\delta}$ estimate
  Byun∼Kim∼Oh, **J. Funct. Anal.**, 2025.

Free transmission problems

Theorem (Jesus, Calc. Var.& PDE., 2022)

- Let $u \in C(B_1)$ be a viscosity solution to

$$|Du|^{\beta(x,u,Du)} F(D^2 u) = f \quad \text{in} \quad B_1,$$

where $F(\cdot)$ is a uniformly elliptic operator, $f \in C(B_1) \cap L^\infty(B_1)$ and $0 < \beta_m \le \beta(\cdot) \le \beta_M < \infty$. Then $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1)$, where

$$\alpha = \min\left\{\alpha_0^-, \frac{1}{1+\beta_M}\right\}.$$

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$$\alpha = \min\left\{\alpha_0^-, \frac{1}{1+\beta_M}\right\}.$$

Remark. Previous strategy cannot be applied directly to this model.

$\leftrightsquigarrow$

- (Optimal pointwise regularity) Let $u \in C(B_1)$ be a viscosity solution to

$$-|Du|^{\beta(x,u,Du)}F(D^2u) = f(x) \ \text{ in } \ B_1,$$

where $\beta(x,u,Du) = \sum_{i=0}^{N} \beta_i(x)\chi_{G_i(u,Du)}(x)$, $\{G_i(u,Du)\}_{i=1}^{N}$ are disjoint sets in B_1, $G_0(u,Du) := B_1 \setminus \cup_{i=1}^{N} G_i$, $\beta_m \leq \beta_i(\cdot) \leq \beta_M$, $i = 0, 1, \cdots, N$, and $f \in C(B_1) \cap L^\infty(B_1)$. Assume also $\{\beta_i(x)\}_{i=0}^{N}$ have modulus of continuity ω satisfying

$$\limsup_{t\to 0} \ln(t^{-1})\omega(t) = 0.$$

Then for every $x_0 \in B_{1/2}$, it holds u is $C^{1,\alpha}(x_0)$, and

$$||u||_{C^{1,\alpha}(x_0)} \leq C\left(||u||_{L^\infty(B_1)} + ||f||_{L^\infty(B_1)}\right),$$

where

$$\alpha = \min_{i=0,1,\cdots,N}\left\{\alpha_0^-, \frac{1}{1+\beta_i(x_0)}\right\}. \leftrightsquigarrow \text{Approximation method.}$$

A classical model

Theorem (Huaroto∼Pimentel∼Rampasso∼Świech, Ann. PDE., 2024)

Suppose $N = 2$, $G_1(u, Du) = \{u > 0\}$, $G_2(u, Du) = \{u < 0\}$, and $G_0(u, Du) = \{u = 0\}$. Additionally, $\beta_0 = 0$, β_1 and β_2 are constants, i.e.,

$$\begin{cases} |Du|^{\beta_1} F(D^2 u) = f & \text{in } B_1 \cap \{u > 0\} \\ |Du|^{\beta_2} F(D^2 u) = f & \text{in } B_1 \cap \{u < 0\}. \end{cases} \tag{FTP}$$

Then $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1)$, and $\alpha = \min\{\alpha_0^-, \frac{1}{1+\max(\beta_1,\beta_2)}\}$.

Very recently, Pimentel∼Sousa(**arXiv:2506.02595.**, 2025) propose finite difference methods for (**FTP**) and prove the convergence of the schemes.

Question

For the general degenerate normalized $p-$Laplacian equation,

$$-\left\{|Du|^{\alpha(x,u)}+a(x)|Du|^{\beta(x,u)}\right\}\Delta_p^{\mathrm{N}} u=f(x) \text{ in } B_1, \qquad \textbf{(G-DNP)}$$

where $0\leq a(x)\in C(B_1)$, $0<a_1\leq\alpha(x,u)\leq\beta(x,u)\leq a_2<\infty$, $1<p<\infty$ and

$$\Delta_p^{\mathrm{N}} u:=\Delta u+(p-2)|Du|^{-2}\langle D^2uDu,Du\rangle.$$

1. Can we obtain the $C^{1,\alpha}$ regularity of the solution?
2. If the degenerate rates $\alpha(x,u)$ and $\beta(x,u)$ are varied over the domain, does the regularity of solutions also depend on the partition of domains?

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where $0 \leq a(x) \in C(B_1)$, $0 < a_1 \leq \alpha(x,u) \leq \beta(x,u) \leq a_2 < \infty$, $1 < p < \infty$ and

$$\Delta_p^{\mathrm{N}} u := \Delta u + (p-2)|Du|^{-2}\langle D^2 u Du, Du\rangle.$$

1. Can we obtain the $C^{1,\alpha}$ regularity of the solution?
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Two difficulities

- The scaling process is more tricky;
- Although such the operator Δ_p^{N} has a uniformly elliptic structure, previous strategy cannot be applied directly since the discontinuity at the set $\{Du = 0\}$.

Outline

Theorem 1 (W.∼Yin∼Jiang, Potential Anal., 2025)

Let $u \in C(B_1)$ be a viscosity solution to (**G-DNP**), then there exists $\alpha' = \alpha'(p, n, a_1, a_2) \in \left(0, \frac{1}{1+a_2}\right]$ such that for any ball $B \Subset B_1$, we have $u \in C^{1,\alpha'}(B)$ with the estimate

$$||u||_{C^{1,\alpha'}(B)} \lesssim \left(||u||_{L^\infty(B_1)} + \max\{||f||_{L^\infty(B_1)}, ||f||_{L^\infty(B_1)}^{\frac{1}{1+a_1}}, ||f||_{L^\infty(B_1)}^{\frac{1}{1+a_2}}\}\right).$$

Remark 1. We only need to assume the positive lower and upper bounds of $\alpha(x,u)$ and $\beta(x,u)$. No continuity requirements of $\alpha(x,u)$ and $\beta(x,u)$ are needed.

Remark 2. The method here does not work for singular equation.

Proof of Theorem 1

Step 1: We consider the existence of viscosity solutions to a Dirichlet problem associated with the anisotropic free transmission problems

$$\begin{cases} -\Big\{|Du|^{a_1\chi_{\{u>0\}}+a_2\chi_{\{u<0\}}} + a(x)\chi_{\{u>0\}}|Du|^{a_1} \\ \qquad\qquad + a(x)\chi_{\{u<0\}}|Du|^{a_2}\Big\}\Delta_p^{\mathrm{N}} u = f(x) & \text{in } B_1 \\ u = g & \text{on } \partial B_1, \end{cases} \tag{2}$$

where a_1, a_2 are nonnegative constants, $0 \leq a(x) \in C(B_1)$, $f(x) \in C(B_1) \cap L^\infty(B_1)$ and $g \in C(\partial B_1)$.

▶ Perron's method and fixed point argument ⇒ existence of viscosity solution to (2);

▶ Viscosity solution $u \in C(B_1)$ of (2) turns out to be a viscosity sub-solution and viscosity super-solution to

$$\min_{i=0,1,2} \left\{ -|Du|^{a_i} \Delta_p^{\mathrm{N}} u \right\} = ||f||_{L^\infty(B_1)} \tag{3}$$

and

$$\max_{i=0,1,2} \left\{ -|Du|^{a_i} \Delta_p^{\mathrm{N}} u \right\} = -||f||_{L^\infty(B_1)}, \tag{4}$$

respectively, where $a_0 = 0$.

Step 2: Based on these two viscosity inequalities (3) and (4), we want to show $u \in C^{1,\alpha'}_{\text{loc}}(B_1)$ for some $\alpha' \in (0,1)$ and $\forall\ B \Subset B_1$,

$$||u||_{C^{1,\alpha'}(B)} \lesssim \Big(||u||_{L^\infty(B_1)} + \max\{||f||_{L^\infty(B_1)}, ||f||^{\frac{1}{1+a_1}}_{L^\infty(B_1)}, ||f||^{\frac{1}{1+a_2}}_{L^\infty(B_1)}\}\Big).$$

[Geometric tangential methods&Improvement of flatness lemma]

Step 2.1: For this purpose, we first show that Hölder regularity of solution u for perturbed equation

$$\min\Big\{-\Delta^{\mathrm{N}}_{p,\xi}u, -|Du+\xi|^{a_1}\Delta^{\mathrm{N}}_{p,\xi}u, -|Du+\xi|^{a_2}\Delta^{\mathrm{N}}_{p,\xi}u\Big\} = ||f||_{L^\infty(B_1)} \quad (5)$$

and

$$\max\Big\{-\Delta^{\mathrm{N}}_{p,\xi}u, -|Du+\xi|^{a_1}\Delta^{\mathrm{N}}_{p,\xi}u, -|Du+\xi|^{a_2}\Delta^{\mathrm{N}}_{p,\xi}u\Big\} = -||f||_{L^\infty(B_1)}, \quad (6)$$

where ξ is an arbitrary vector in $\mathbb{R}^n$ and

$$\Delta^{\mathrm{N}}_{p,\xi}u = \Delta u + (p-2)\Big\langle D^2u\frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|}\Big\rangle, \quad 1<p<\infty.$$

▶ Whenever $|\xi|$ is large, Ishii-Lions method $\Rightarrow u \in C^{0,1}_{\text{loc}}(B_1)$;
▶ Whenever $|\xi|$ is small, the result of [Imbert∼Silvestre, **J. Eur. Math. Soc.**, 2016] $\Rightarrow u \in C^{0,\beta}_{\text{loc}}(B_1)$ for some $\beta \in (0,1)$.

Step 2.2:

Improvement of flatness lemma

Suppose u is a viscosity sub-solution to (5) and u is viscosity super-solution to (6). Then there exists $0 < \rho < 1$ and $\delta > 0$, depending only on p, n and a_2, such that if $||u||_{L^\infty(B_1)} \leq 1/2$ and $||f||_{L^\infty(B_1)} \leq \delta$, the inequality

$$\underset{B_\rho}{osc}\left(u - \vec{q}\cdot x\right) \leq \frac{1}{2}\rho$$

holds for some $\vec{q} \in \mathbb{R}^n$.

Remark. For its proof, cutting lemma, the compactness of solution and $C^{1,\beta_1}_{\text{loc}}$ regularity for homogeneous normalized $p-$Laplacian equation are essential.

$$-|Du|^{a_1}\Delta_p^{\mathrm{N}} u = 0 \quad \text{in} \quad B_1 \quad \Rightarrow \quad \Delta_p^{\mathrm{N}} u = 0 \quad \text{in} \quad B_1.$$

Lemma 4.3 *Let v be a viscosity solution of*

$$-\Delta v - (p-2)\left\langle D^2 v \frac{Dv+\xi}{|Dv+\xi|}, \frac{Dv+\xi}{|Dv+\xi|} \right\rangle = 0 \text{ in } B_1, \quad \xi \in \mathbb{R}^n,$$

with $||v||_{L^\infty(B_1)} \leq \frac{1}{2}$. For all $0 \leq r \leq \frac{1}{2}$, there exists constants $C_0 = C_0(p,n) > 0$ and $\beta_1 = \beta_1(p,n) > 0$ such that

$$||v||_{C^{1,\beta_1}(B_r)} \leq C_0.$$

Figure 1: $C^{1,\beta_1}_{\text{loc}}$ regularity for homogeneous normalized $p-$Laplacian equation

Step 2.3: Standard iteration $\Rightarrow u \in C^{1,\alpha'}_{\text{loc}}(B_1)$

Step 3: The definition of viscosity solution $\Rightarrow$ the solution to (**G-DNP**) fulfills these two viscosity inequalities (3) and (4).

Theorem 2 (W.~Yin~Jiang, **Potential Anal.**, 2025)

Let $u \in C(B_1)$ be a bounded viscosity solution to (**G-DNP**), where

- $\Omega_i(u) \subset B_1, i = 1, 2, \cdots, M$ be disjoint sets, and $\Omega_0(u) := B_1 \setminus \cup_{i=1}^{M} \Omega_i$;
-
$$\alpha(x,u) = \sum_{i=0}^{M} \alpha_i(x)\chi_{\Omega_i(u)}, \quad \beta(x,u) = \sum_{i=0}^{M} \beta_i(x)\chi_{\Omega_i(u)}$$
- there is a non-decreasing function $\omega : [0, +\infty) \to [0, +\infty)$ such that
$$|\alpha_i(x) - \alpha_i(y)| + |\beta_i(x) - \beta_i(y)| \le \omega(|x-y|), \ \ i = 0, 1, \cdots, M$$
and ω satisfies the balancing condition
$$\limsup_{t\to 0} \omega(t)\ln(t^{-1}) = 0.$$
Then for every $x_0 \in B_{1/2}$, we have that u is $C^{1,\tau}(x_0)$, and

$\rightsquigarrow$

$$||u||_{C^{1,\tau}(x_0)} \lesssim \left(1 + ||u||_{L^\infty(B_1)} + ||f||^{\gamma}_{L^\infty(B_1)}\right),$$

where

$$\tau = \min_{i=0,1,\cdots,M} \left\{ \widehat{\beta_0}^{\,-}, \frac{1}{1+\alpha_i(x_0)} \right\}, \quad \gamma = \frac{1}{1 + \min\limits_{i=0,1,\cdots,M} \inf_{B_1} \alpha_i(x)}.$$

Remark 3. $\widehat{\beta_0}$ is the nearly optimal Hölder exponent to the gradient for solutions of the normalized p-harmonic function.

Remark 4. Our argument is based on a new improved oscillation-type estimate combined with a localized analysis, which is totally different from the approach adopted by Jesus.

Proof of Theorem 2

Step 1: Reduction of the problem

Suppose u is a viscosity solution to (**G-DNP**), then u is a viscosity sub-solution to

$$\min_{i=0,1,\cdots,M}\left\{-\left(|Du|^{\alpha_i(x)}+a(x)|Du|^{\beta_i(x)}\right)\Delta_p^{\mathrm{N}}u\right\}=||f||_{L^\infty(B_1)} \tag{7}$$

and is also a viscosity super-solution to

$$\max_{i=0,1,\cdots,M}\left\{-\left(|Du|^{\alpha_i(x)}+a(x)|Du|^{\beta_i(x)}\right)\Delta_p^{\mathrm{N}}u\right\}=-||f||_{L^\infty(B_1)}. \tag{8}$$

A simple scaling reduces the proof of the problem to the case that

$$||u||_{L^\infty(B_1)}\le 1 \;\text{ and }\; ||f||_{L^\infty(B_1)}\le \delta$$

for some positive constant δ to be determined.

Step 2: Approximation

Proposition 1

Assume (7)–(8) hold. For given $0 < \epsilon < 1$, there exists $\delta > 0$, depending only on n, p, a_1, a_2, such that if $||u||_{L^\infty(B_1)} \leq 1$ with $||f||_{L^\infty(B_1)} \leq \delta$, then there exists a function v satisfying

$$-\Delta_p^{\mathrm{N}} v = 0 \ \text{ in } \ B_r \ (0 < r < 1)$$

in the viscosity sense, and

$$||u - v||_{L^\infty(B_r)} \leq \epsilon \ \text{ and } \ ||Du - Dv||_{L^\infty(B_r)} \leq \epsilon.$$

Step 3: Improved oscillation-type estimate

Proposition 2

Under the assumptions in Proposition 1, and for every $0 < \tau' < \widehat{\beta}_0$, then there exists universal constant $0 < \rho \ll 1$ such that

$$\sup_{B_\rho(0)} |u(x) - u(0)| \leq \rho^{1+\tau'} + |\mathbf{Du(0)}|\rho.$$

By iterating the oscillation estimate above, we obtain

$$\sup_{B_{\rho^k}(0)} |u(x) - u(0)| \leq \rho^{k(1+\tau_k)} + |\mathbf{Du(0)}| \sum_{\mathbf{i=0}}^{\mathbf{k-1}} \rho^{\mathbf{k+i\tau_k}} \tag{9}$$

for a non-decreasing sequence $\{\tau_k\}$.

Step 4: Localized analysis

▶ Whenever $|Du(0)|$ is small, by using improved oscillation-type estimate (9) in Proposition 2, we obtain that u is $C^{1,\tau}(0)$, where

$$\tau = \min_{i=0,1,\cdots,M} \left\{ \widehat{\beta_0}^{-}, \frac{1}{1+\alpha_i(0)} \right\}.$$

Recall that $\widehat{\beta_0}$ is the nearly optimal Hölder exponent to the gradient for solutions of $-\Delta_p^{\mathrm{N}} u = 0$. (Attouchi∼Parviainen∼Ruosteenoja, **J. Math. Pures Appl.**, 2017)

▶ Whenever $|Du(0)|$ is large, then the equation becomes non-degenerate and classical estimate (Caffarelli's regularity theory) can apply.

Part II: Degenerate/Singular fully nonlinear non-local equations

Consider

$$-|Du|^{\gamma}\mathcal{I}_{\sigma}(u,x)=f(x) \text{ in } B_1,$$

where $\gamma>0, \sigma\in(0,2)$, and $\mathcal{I}_{\sigma}$ is a fully nonlinear elliptic integro-differential operator

$$\mathcal{I}_{\sigma}(u,x)=\inf_{i}\sup_{j}\left[P.V.\int_{\mathbb{R}^n}(u(y)-u(x))\,K_{ij}(x-y)dy\right], \tag{10}$$

with a two-parameter family of symmetric kernels $\{K_{ij}\}\subseteq\mathcal{K}_0$ and $\mathcal{K}_0$ denotes the family of symmetric kernels $K:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}_+$ satisfying

$$\lambda\frac{C_{n,\sigma}}{|x|^{n+\sigma}}\le K(x)\le\Lambda\frac{C_{n,\sigma}}{|x|^{n+\sigma}},$$

for the order σ and certain constants $0<\lambda\le\Lambda<\infty$. Note that (10) is well defined for function $u\in C^{1,1}(B_{\delta}(x))\cap L^1_{\sigma}(\mathbb{R}^n)$ for some $\delta>0$, where

$$||u||_{L^1_{\sigma}(\mathbb{R}^n)}:=\int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+\sigma}}dx<+\infty.$$

Theorem (Prazeres∼Topp, **J. Differential Equations.**, 2021)

- If $\sigma \in (0,1)$, then $u \in C^{\sigma}_{\text{loc}}(B_1)$;
- If $\sigma = 1$, then $u \in C^{\alpha}_{\text{loc}}(B_1)$ for every $\alpha \in (0,1)$;
- If $\sigma \in (1,2)$, then $u \in C^{0,1}_{\text{loc}}(B_1)$.

The above result results can be obtained by adjusting the auxiliary functions and using Ishii-Lion's method.

Question: How to improve regularity from $C^{0,1}$ to $C^{1,\alpha}$?

Key difficulty : Cutting lemma is not valid.

$$-|Du|^{\gamma}\mathcal{I}_{\sigma}(u,x) = 0 \text{ in } B_1 \;\not\Rightarrow\; \mathcal{I}_{\sigma}(u,x) = 0 \text{ in } B_1.$$

Counter-example

Example

We consider

$$u(x) = \begin{cases} x+1, & \text{if } \ x \le -1, \\ 0, & \text{if } \ -1 < x < 1, \\ x-1, & \text{if } \ x \ge 1, \end{cases}$$

then straightforward computations yield

$$-|u_x|^\gamma \Delta^{\frac{\sigma}{2}} u = 0 \ \text{ in } \ (-1,1), \quad \sigma \in (1,2).$$

Since $\Delta^{\frac{\sigma}{2}} u(x) \to -\infty$, $x \to -1^+$ and $\Delta^{\frac{\sigma}{2}} u(x) \to +\infty$, $x \to 1^-$, hence

$$-\Delta^{\frac{\sigma}{2}} u(x) \neq 0 \ \text{ in } \ (-1,1).$$

Additional condition

(AC): There exists a modulus of continuity ω and $\{k_{ij}\}_{i,j} \subseteq (\lambda, \Lambda)$ such that

$$\left| K_{ij}(x)|x|^{n+\sigma} - k_{ij} \right| \leq \omega(|x|) \text{ for all } i,j \text{ and } |x| \leq 1.$$

Here we provide an explanation.

$$\begin{aligned}
\mathcal{I}_\sigma(u,x) &= \inf_i \sup_j P.V. \int_{\mathbb{R}^n} (u(y) - u(x))\, K_{ij}(x-y)dy \\
&\sim \inf_i \sup_j k_{ij} P.V. \int_{\mathbb{R}^n} \frac{u(y)-u(x)}{|x-y|^{n+\sigma}} C_{n,\sigma} dy \quad \text{(by (AC))} \\
&= \inf_i \sup_j k_{ij} \Delta^{\frac{\sigma}{2}} u \to \inf_i \sup_j k_{ij} \mathrm{Tr}(D^2 u) \quad (\text{as } \sigma \to 2^-).
\end{aligned}$$

$\Rightarrow$

$$\boxed{-|Du|^\gamma \inf_i \sup_j k_{ij} \mathrm{Tr}(D^2 u) = 0 \ \text{ in } \ B_1 \quad \Rightarrow \quad \inf_i \sup_j k_{ij} \mathrm{Tr}(D^2 u) = 0 \ \ in \ \ B_1.}$$

Theorem (Prazeres∼Topp, **J. Differential Equations.**, 2021)

Suppose the condition **(AC)** also holds, then there exists $\sigma_0 \in (1,2)$ close enough to 2 such that for $\sigma_0 < \sigma < 2$ every bounded viscosity solution u is in $C^{1,\alpha}$ for

$$\alpha = \min\left\{\alpha_0^-, \frac{\sigma-1}{1+\gamma}\right\},$$

and

$$||u||_{C^{1,\alpha}(B_{1/2})} \le C\left(||u||_{L^\infty(B_1)} + ||u||_{L^1_\sigma(\mathbb{R}^n)} + ||f||^{\frac{\sigma-1}{1+\gamma}}_{L^\infty(B_1)}\right).$$

Remark. The constant C that appears in the estimate above is uniform in σ, which means that it does not blow up as σ approaches 2.

We consider a series of regularity results for solutions to the double-phase degenerate or singular fully nonlinear integro-differential equation of the form

$$-\big(\sigma_1(|Du|)+a(x)\sigma_2(|Du|)\big)\mathcal{I}_\sigma(u,x)=f(x) \quad \text{in} \quad B_1, \qquad (\mathbf{DPE})$$

We first present a list of appropriate assumptions on σ, u, f, a and σ_i, $i=1,2$.

(A1) (order of the operator). $\sigma\in(1,2)$.

(A2) (growth condition at infinity). $u\in L^1_\sigma(\mathbb{R}^n)$.

(A3) (regularity of source term). $f\in L^\infty(B_1)\cap C^0(B_1)$.

(A4) (nonnegativity and regularity of $a(x)$). $0\le a(x)\in C^0(B_1)$.

(A5) (convergence of the operator). Let $\{K_{ij}\}$ be a collection of kernels of in $\mathcal{K}_0$. There exists a modulus of continuity ω and $\{k_{ij}\}\in[\lambda,\Lambda]$ satisfying

$$|K_{ij}(x)|x|^{n+\sigma}-k_{ij}|\le\omega(|x|) \text{ for all } i,j \text{ and } |x|\le 1.$$

(A6) (monotonicity of $\sigma_i, i = 1, 2$).
- **(A6a) (degenerate case).** $\sigma_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2$, are continuous and monotone increasing with

$$\lim_{t \to 0} \sigma_i(t) = 0, \quad i = 1, 2.$$

- **(A6b) (singular case).** $\sigma_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2$, are continuous and monotone decreasing with

$$\lim_{t \to 0} \sigma_i(t) = c_0 > 0, \quad i = 1, 2.$$

Furthermore, in both cases, we assume

$$\sigma_1(t) \geq \sigma_2(t), \quad t \in [0, 1],$$

and, in particular,

$$\sigma_1(1) \geq \sigma_2(1) \geq 1.$$

(A7) (Dini continuity of the inverse of σ_2). The function σ_2 admits an inverse σ_2^{-1} that is Dini continuous, i.e.,

$$\int_0^1 \frac{\sigma_2^{-1}(t)}{t} dt < \infty.$$

(A8) (Non-collapsing of $\sigma_1 + a(\cdot)\sigma_2$). $\sigma_1 + a(\cdot)\sigma_2 \in \Sigma$, where Σ is a set of non-collapsing moduli of continuity.
A set Σ of moduli of continuity is said to be non-collapsing if for all sequences $(f_k)_{k\in\mathbb{N}} \subset \Sigma$, and all sequences of $(a_k)_{k\in\mathbb{N}} \subset (0,\infty)$, we have

$$\lim_{k\to+\infty} f_k(a_k) = 0 \quad \Rightarrow \quad \lim_{k\to+\infty} a_k = 0.$$

Theorem 1 (W.~Jiang, **Commun. Contemp. Math.**, 2025)

[Borderline regularity in the degenerate case]
Let $u \in C^0(\overline{B}_1)$ be a bounded viscosity solution to (**DPE**), assume **(A1)–(A5)**, **(A6a)** and **(A7)** hold. Then there exists $\sigma_0 \in (1,2)$ sufficiently close to 2 such that if $\sigma \in (\sigma_0, 2)$, then $u \in C^1_{loc}(B_1)$. Moreover, there exists a modulus of continuity $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ depending only on $n, \sigma, \sigma_1, \sigma_2, ||u||_{L^\infty(B_1)}, ||f||_{L^\infty(B_1)}$ such that

$$|Du(x) - Du(y)| \leq \omega(|x-y|),$$

for every $x, y \in B_{1/4}$. In particular, if $\sigma_i^{-1} (i = 1, 2)$ behave like Hölder continuous functions near the origin, then the solutions are locally $C^{1,\alpha}$, for some $0 < \alpha < 1$.

Remark 1. Note that it suffices to require σ_2^{-1} to be a Dini-continuous modulus of continuity, without similar assumption on σ_1^{-1}. Indeed, σ_1^{-1} is automatically Dini-continuous since σ_2^{-1} is Dini-continuous and $\sigma_1 \geq \sigma_2$.

Remark 2. Theorem 1 embraces the main result of (Andrade∼Pellegrino∼Pimentel∼Teixeira, **Adv. Math.**, 2022).

Remark 3. the main difficulty is that the degenerate term $\sigma_1 + a(\cdot)\sigma_2$ lacks of *non-collapsing* property. To handle such a challenge, we need to establish a new *shored up*-continuous module sequence $(\sigma_1^n)_{n\in\mathbb{N}}$.

Definition

A sequence of moduli of continuity $(\sigma_1^k)_{k\in\mathbb{N}}$ is said to be shored-up if there exists a sequence of positive numbers $(c_k)_{k\in\mathbb{N}}$ such that $c_k \to 0$ satisfying $\inf_k \sigma_1^k(c_k) > 0$ for every $k \in \mathbb{N}$.

Lemma

If a sequence of moduli of continuity $(\sigma_1^k)_{k\in\mathbb{N}}$ is shored-up, then $\Gamma := \cup_{k\in\mathbb{N}}\{\sigma_1^k\}$ is non-collapsing.

We consider higher regularity of solutions for

$$-\left(|Du|^{\widetilde{p}}+a(x)|Du|^{q}\right)\mathcal{I}_{\sigma}(u,x)=f(x)\ \text{ in }\ B_1, \tag{11}$$

where $0<\widetilde{p}\leq q<\infty$, and $\widetilde{p}<\sigma-1$. Here we give source term f a stronger assumption than **(A3)**, namely, **(A3a)**: suppose that $f(0)=0$ and for some $K>0$ and $\gamma\in(\widetilde{p}+2-\sigma,1)$, there holds $|f(x)|\leq K|x|^{\gamma}$.

Theorem 2 (W.∼Jiang, **Commun. Contemp. Math.**, 2025)

[Higher regularity in the degenerate case]

Let $u\in C^0(\overline{B}_1)$ be a bounded viscosity solution to (11), assume **(A1)**, **(A2)**, **(A3a)**, **(A4)**and **(A5)** hold. Assume in addition that 0 is a local extrema for u. Then, there exists $\sigma_0\in(1,2)$ sufficiently close to 2 such that if $\sigma\in(\sigma_0,2)$, then $u\in C^{2,\alpha}(0)$ and

$$|u(x)-u(0)|\leq C|x|^{2+\alpha},$$

for any $x\in B_{1/10}(0)$, where $\alpha=\frac{\gamma+\sigma-2-\widetilde{p}}{1+\widetilde{p}}\in(0,1)$, and $C=C(n,\gamma,\sigma,\widetilde{p},\lambda,\Lambda)>0$ is a positive constant.

Remark 4. Observing the constant C in Theorem 2 will not blow up as $\sigma \to 2$. Hence, Theorem 2 could be considered as an extension of (Nascimento, **J. Funct. Anal.**, 2025).

Theorem 3 (W.∼Jiang, **Commun. Contemp. Math.**, 2025)

[C^{1,β_0} regularity in the singular case]
Let $u \in C^0(\overline{B}_1)$ be a bounded approximated viscosity solution to (**DPE**), assume **(A1)**, **(A2)**, **(A3)**, **(A4)** and **(A6b)** hold. Then $u \in C^{1,\beta_0}_{\text{loc}}(B_1)$. Moreover, there exists $C > 0$ depending only on $n, \lambda, \Lambda, ||f||_{L^\infty(B_1)}$ and $||u||_{L^\infty(B_1)}$ such that

$$|Du(x) - Du(y)| \leq C|x - y|^{\beta_0},$$

for some $\beta_0 \in (0, 1)$ and every $x, y \in B_{1/2}$.

Remark 5. The condition **(A6b)** also contains some important examples, for instance,

$$\sigma_1(t) = \frac{\ln^\beta(1+t)+1}{t}, \quad \sigma_2(t) = \frac{\ln^\beta(1+t)}{t}, \quad 0 < \beta \leq 1.$$

and

$$\sigma_1(t) = \frac{(e^t - 1)^\beta + 1}{t}, \quad \sigma_2(t) = \frac{(e^t - 1)^\beta}{t}, \quad 0 < \beta \leq 1 - \frac{1}{e}.$$

Remark 6. In the singular case, the difficulty starts with the notion of viscosity solution. Indeed, the definition considered in the local case seems not to be suitable in our scenario, because whenever $x_0 \in \{Du = 0\}$, we have that $u \equiv c$ in $B_{r(x_0)}(x_0)$, which implies that $F(D^2u) = 0$ in $B_{r(x_0)}(x_0)$ and hence one could say that $0 \leq f(x_0)$(in the case of supersolutions).

On the other hand, in the nonlocal case, we still need to consider the quantity

$$\int_{\mathbb{R}^n \setminus B_{r(x_0)}(x_0)} (u(x_0) - u(y)))K(y)dy,$$

which depends on $r(x_0)$ and may not vanish. To overcome this, we use the notion of approximated viscosity solution, which coincides with the usual notion of viscosity solution over the set $\{Du \neq 0\}$.

Outline

Parabolic dead-core problems

We study geometric regularity estimates for degenerate fully nonlinear parabolic equations of the form

$$|Du|^p F(D^2 u) - u_t = \lambda_0(x,t)\, u^\mu \chi_{\{u>0\}}(x,t) \quad \text{in} \quad Q_T := Q \times (0,T), \qquad \textbf{(DCP)}$$

where

- $0 \le p < \infty$, $0 < \mu < 1$, $T > 0$, and $Q \subset \mathbb{R}^n$ is a bounded smooth domain;
- $0 < \mathcal{M}_1 \le \lambda_0(x,t) \le \mathcal{M}_2 < \infty$.

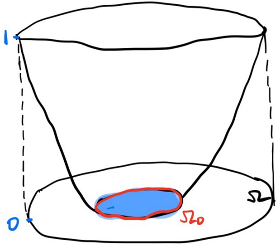
The operator $F : \mathrm{Sym}(n) \to \mathbb{R}$ satisfies the following structural conditions:

(F1) F is uniformly elliptic with $F(\mathrm{O}_n) = 0$;

(F2) F is convex or concave;

(F3) $F \in C^{1,\kappa}$ for some $\kappa \in (0,1]$.

Dead-core phenomenon



Ω_0 : dead-core domain.

Classical Model

$$\begin{cases} -\Delta u(x) + \lambda_0(x)\, u_+^{q}(x) = 0 & \text{in } \Omega, \\ u(x) = 1 & \text{on } \partial\Omega. \end{cases}$$

$0 < q < 1$ & $\lambda_0 > 0$.

u：化学试剂（气体）的密度；

λ_0：蒂勒模数，控制反应速率与扩散-对流速率的比值.

Theorem 1 (da Silva∼Jiang∼W., **arXiv:2512.08196.**, 2025)

[Improved regularity along the free boundary]
Suppose that F satisfies **(F1)**, **(F2)** and **(F3)**. Let u be a nonnegative and bounded viscosity solution to (**DCP**), so that $\partial_t u \geq 0$ and $K \Subset Q_T$ a compact set. Then there exists a constant $\mathrm{C}_0 > 0$, depending only on $n, \lambda, \Lambda, \mathcal{M}_1, \mathcal{M}_2, p, \mu$ and $\mathrm{dist}(K, \partial_{par} Q_T)$ such that for all $(x_0, t_0) \in \partial\{u > 0\} \cap K$,

$$u(x,t) \leq \mathrm{C}_0 \|u\|_{L^\infty(Q_T)} \mathrm{dist}_p((x,t),(x_0,t_0))^{\frac{2+p}{1+p-\mu}},$$

for all (x,t) sufficiently close to (x_0,t_0), where

$$\mathrm{dist}_p((x,t),(x_0,t_0)) = |x - x_0| + |t - t_0|^{\frac{1}{\theta}}, \quad \textit{and} \quad \theta = \frac{(2+p)(1-\mu)}{1+p-\mu}.$$

Remark 1. The proof of Theorem 1 involves the construction of a delicate barrier function defined by

$$\Phi(x,t) = \mathcal{C}\left(\mathcal{A}|x|^{\frac{2+p}{1+p}} + \mathcal{B}t^{\frac{1+p-\mu}{(1+p)(1-\mu)}}\right)^{\frac{1+p}{1+p-\mu}},$$

for universal constants $\mathcal{C}, \mathcal{A}, \mathcal{B} > 0$, together with an application of a comparison principle. We remark that the condition $0 < \mu < 1$ is essential to prevent the barrier function $\Phi(x,t)$ from blowing up as $t \to 0^+$.

Remark 2. The solutions to (**DCP**) have a refined point-wise behaviour as follows:

$$\sup_{Q_r(x_0,t_0)} u(x,t) \lesssim r^{\frac{2+p}{1+p-\mu}}$$

along the free boundary. Theorem 1 unveils that the solutions are expected to be $C^{\left\lfloor \frac{2+p}{1+p-\mu} \right\rfloor, \frac{\left\lfloor \frac{2+p}{1+p-\mu} \right\rfloor}{\theta}}$ at the free boundary points, where

$$\theta := \theta(p, \mu) := \frac{(2+p)(1-\mu)}{1+p-\mu} = 1 + \frac{1-\mu(p+1)}{p+1-\mu} \leq 2.$$

We point out that for $p \geq 0$:

$$\theta(p, \mu) := \frac{(2+p)(1-\mu)}{1+p-\mu} > 0 \quad \Leftrightarrow \quad \mu < 1,$$

which validates that assumption $0 < \mu < 1$ is very necessary. From [Lee∼Lian∼Yun∼Zhang, **Preprint**, 2025], we obtain $C^{1,1/(1+p)}$ spatial regularity. Then we can observe that for $0 \leq p < \infty$, $0 < \mu < 1$,

$$\frac{2+p}{1+p-\mu} > 1 + \frac{1}{1+p} \quad \Leftrightarrow \quad \mu > 0.$$

Thus, we derive better regularity estimates along $\partial\{u > 0\} \cap Q_T$ than the regularity available currently([Lee∼Lian∼Yun∼Zhang, **Preprint**, 2025], [Lee∼Lee∼Yun, **J. Math. Pures Appl.**, 2024]).

Remark 3. The method here does not work for $-1 < p < 0$.

Theorem 2 (da Silva∼Jiang∼W., **arXiv:2512.08196.**, 2025)

[Non-degeneracy]

Suppose F satisfies **(F1)**. Let u be a viscosity solution to (**DCP**). Then for every $(x_0, t_0) \in \overline{\{u > 0\}}$ and $Q_r(x_0, t_0) \Subset Q_T$, there holds

$$\sup_{\partial_{par} Q_r^-(x_0,t_0)} u(x,t) \geq \mathrm{C}_0^* r^{\frac{2+p}{1+p-\mu}}$$

for a positive constant $\mathrm{C}_0^* = \mathrm{C}_0^*(n, \lambda, \Lambda, \mathcal{M}_1, \mathcal{M}_2, p, \mu)$.

Corollary 1

Suppose that F satisfies **(F1)**, **(F2)** and **(F3)**. Let u be a nonnegative, bounded viscosity solution to (**DCP**) and $Q' \Subset Q_T$. For given $(x_0, t_0) \in \{u > 0\} \cap Q'$, there exist universal constants $\mathrm{C}^* > 0$ and $\mathrm{C}_* > 0$ such that

$$\mathrm{C}_* \mathrm{dist}_p((x_0, t_0), \partial\{u > 0\})^{\frac{2+p}{1+p-\mu}} \leq u(x_0, t_0) \leq \mathrm{C}^* \mathrm{dist}_p((x_0, t_0), \partial\{u > 0\})^{\frac{2+}{1+p}}$$

Corollary 2

[Positive Lebesgue density of $\{u > 0\}$]
Suppose that F satisfies **(F1)**, **(F2)** and **(F3)**. Let u be the viscosity solution to (**DCP**). Then, there exists a positive constant $\zeta = \zeta(n, \lambda, \Lambda, \mathcal{M}_2, \|u\|_{L^\infty(Q_1)})$ such that for all $(x_0, t_0) \in \overline{\{u > 0\}}$ and $0 < r < 1$ such that $Q_r(x_0, t_0) \subset Q_{1/2}$, the inclusion

$$Q_{\zeta r}(\widetilde{x}, \widetilde{t}) \subset Q_r(x_0, t_0) \cap \{u > 0\}$$

holds for some $(\widetilde{x}, \widetilde{t}) \in Q_r^-(x_0, t_0)$.

We observe that Corollary 2 guarantees that the free boundary cannot have Lebesgue points. In other words, $\mathcal{L}^{n+1}(\partial\{u>0\}\cap K)=0$ for any compact set $K\subset Q_1$.

Definition (**Porous set**)

A set $E\subset\mathbb{R}^n$ is said to be *porous* with porosity constant $0<\delta\leq 1$ if there exists $R>0$ such that, for each $x_0\in E$ and $0<r<R$, there exists a point y_0 satisfying

$$B_{\delta r}(y_0)\subset B_r(x_0)\setminus E.$$

We observe that a porous set has Hausdorff dimension at most $n-c_0\,\delta^n$, where $c_0=c_0(n)>0$.

Corollary 3

[Porosity of $t-$level free boundaries]

Suppose that F satisfies **(F1)**, **(F2)** and **(F3)**. Let u be a viscosity solution to (**DCP**). For every compact set $K\Subset Q_1$, one has

$$H^{n-\varepsilon}\left(\partial\{u>0\}\cap K\cap\{t=t_0\}\right)<\infty,\text{for some constant }\varepsilon>0.$$

Theorem 3

[Gradient decay]

Suppose that F satisfies **(F1)**, **(F2)**,**(F3)** and $0 < \mu \leq \frac{1}{1+p}$ also holds. Let u be a continuous viscosity solution to (**DCP**). For any $(z, s) \in \{u > 0\} \cap Q_{1/2}$, there holds

$$|Du(z,s)| \leq \mathrm{C}\, \mathrm{dist}_p\big((z,s), \partial\{u > 0\}\big)^{\frac{1+\mu}{1+p-\mu}},$$

where $\mathrm{C} > 0$ is a universal constant.

Theorem 4

[Liouville-type result]

Suppose that F satisfies **(F1)**, **(F2)** and **(F3)**. Let u be an entire viscosity solution to

$$|Du|^p F(D^2u) - \partial_t u = \lambda_0(x,t)u^{\mu}\chi_{\{u>0\}}(x,t)$$

with $u(0,0)=0$ and λ_0 as before. If $u(x,t) = o\Big(\max\{|x|,|t|^{\frac{1}{\theta}}\}^{\frac{2+p}{1+p-\mu}}\Big)$ as $\max\{|x|,|t|^{\frac{1}{\theta}}\} \to \infty$, then

$$u \equiv 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}.$$

Question: How to establish Hausdorff measure estimates of the free boundary?

More results on dead-core problems

- Fully nonlinear elliptic equation
  Teixeira, **Math. Ann.**, 2016;
- Fully nonlinear elliptic system
  Araújo∼Teymurazyan, **J. Funct. Anal.**, 2024;
- Degenerate or Singular fully nonlinear elliptic systems
  W.∼Jiang, **arXiv:2502.10099.**, 2025.
- Infinity Laplacian equation
  Araújo∼Leitão∼Teixeira, **J. Funct. Anal.**, 2016.
- Grad-Mercier equation
  Caffarelli∼Farahão∼Restrepo, **arXiv:2504.12548.**, 2025.

Outline

▶ Consider Neumman/Oblique boundary value problem for degenerate fully nonlinear non-local equations, how to establish their sharp boundary $C^{1,\alpha}$ regularity?

[Araújo∼dos Prazeres∼Topp, **arXiv:2306.15452.**, 2023.]

- Feldman∼Huang, Regularity theory of a gradient degenerate Neumann problem, **JMPA.**, 2026.
- Feldman∼Huang, Homogenization of a vertical oscillating Neumann condition, **ARMA.**, to appear.

▶ Consider the obstacle problem for degenerate fully nonlinear elliptic equation, how to establish the regularity of the free boundary?

[da Silva∼Vivas, **Rev. Mat. Iberoam.**, 2021.]

- Ki-Ahm Lee, *Obstacle problems for the fully nonlinear elliptic operators*. Thesis (Ph.D.)–New York University. 1998. 53 pp.

Thanks for your attention.