Fractional Laplacian & Fractional Sobolev Space.

$(-\Delta)^s$.

1. Fourier multiplier. $\widehat{(-\Delta)^s f}(\xi) = |\xi|^{2s}\hat{f}(\xi)$

2. Heat semigroup. $(-\Delta)^s f(x) = \frac{1}{\Gamma(-s)}\int_0^\infty (e^{t\Delta}f(x) - f(x))\frac{1}{t^{1+s}}dt$

* 3. $(-\Delta)^s f(x) = C_{n,s}\int_{\mathbb{R}^n}\frac{f(x)-f(y)}{|x-y|^{n+2s}}dy = C_{n,s}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon}\frac{f(x)-f(y)}{|x-y|^{n+2s}}$

4. Functional Calculus.

$-\Delta$ self-adjoint. $\Rightarrow$ can take any $F:\mathbb{R}^+\to\mathbb{R}$.

consider $F(-\Delta)$.

5. Generator of Levy process.

$$(-\Delta)^{\alpha/2}f(x) = \lim_{h\to 0^+}\frac{1}{h}\mathbb{E}[f(x) - f(x+X_h)].$$

X isotopic α-stable Levy process.

Extension (Caffarelli - Silvestre '07).

$x\in\mathbb{R}^{n+1}$, $y\in\mathbb{R}$, $(x',y)=x$

Dirichlet to Neumann map

$u\in W^{2,s}(\mathbb{R}^n)$. let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ st.

$$\begin{cases}\operatorname{div}(y^a\nabla f)=0 & \text{in } y>0\\ f(x,0)=u\end{cases}$$

here $a=1-2s$.

Then $C(-\Delta)^s u = \lim_{y\to 0} -y^a f_y = \frac{1}{1-a}\lim_{y\to 0}\frac{f(x,y)-f(x,0)}{y^{1-a}}$.

$f_y = \frac{\partial f}{\partial y}$

Rmk. $(-\Delta)^s$ is nonlocal. at any pt x, $(-\Delta)^s u(x)$ "see" all values of u in $\mathbb{R}^n$.

Dirichlet problem.

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega. \\ u = g. & \text{in } \mathbb{R}^n \setminus \Omega \quad (\text{boundary condition}). \end{cases}$$

If $(-\Delta)^s u = 0 \Rightarrow u \in C^\infty$ in the interior.

$u = g$ on $\mathbb{R}^n \setminus \Omega$. $d = dist(x, \partial\Omega)$. $\partial\Omega \in C^{2,\alpha}$. $g \in C^\alpha$.

then. $\frac{1}{d^s} u \in C^{\alpha+s}$.

Natural space : $H^s = W^{s,2}$

Integration by part :

$$\langle u, v \rangle = \frac{1}{2} \iint_{\mathbb{R}^n \mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx dy.$$

For $u, v \in C_c^\infty(\mathbb{R}^n)$. then

$$\int_{\mathbb{R}^n} u (-\Delta)^s v = \langle u, v \rangle = \int_{\mathbb{R}^n} (-\Delta)^s u \cdot v.$$

Pf. 1. $\langle u, v \rangle$ well defined.

$$|\langle u, v \rangle| \le \frac{1}{4} \iint \frac{|u(x+y) - u(y)|^2}{|x-y|^{n+2s}} + \frac{1}{4} \iint \frac{|v(x+y) - v(y)|^2}{|x-y|^{n+2s}}$$

Let $Supp(u) \subset B_R$.

$$|u(x+y) - u(y)|^2 \le C \|u\|_{C^2(\mathbb{R}^n)} \min\{1, |x|^2\}. \ \chi_{B_R \cup B_R(-x)}(y)$$

$$\Rightarrow \langle u, v \rangle < \infty$$

2. Consider $B_\varepsilon^c \times \mathbb{R}^n$

$$\iint_{B_\varepsilon^c \times \mathbb{R}^n} \frac{(u(x+y)-u(x))(v(x+y)-v(x))}{|y|^{n+2s}}$$

$$= \iint \frac{u(x+y)(v(x+y)-v(x)) - u(x)(v(x+y)-v(x))}{|y|^{n+2s}}$$

$y \mapsto y-x$.

$$= \iint \frac{u(x)(2v(y)-v(y-x)-v(y+x))}{|x-y|^{n+2s}}.$$

take $\varepsilon \to 0$. $\quad = 2\int_{\mathbb{R}^n} u(-\Delta)^s v$, $\quad \square$.

Rmk. We need regularity for u, v for $\langle u, v \rangle$.

Let $H^s(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n : \|u\|_{H^s} < \infty\}$.

$$\|u\|_{H^s(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + [u]_{H^s(\mathbb{R}^n)}$$

$$[u]^2_{H^s(\mathbb{R}^n)} = \iint_{\mathbb{R}^n \mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}. \quad \leftarrow \text{Gagliardo "semi" norm.}$$

Fractional Sobolev inequality

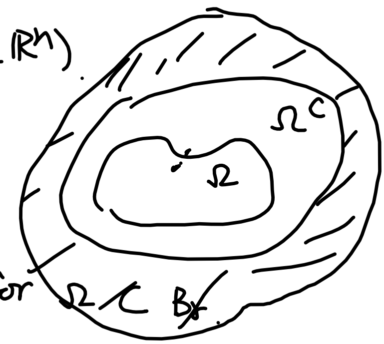
$u \in H^s(\mathbb{R}^n)$, $n > 2s$

$$\|u\|_{L^q(\mathbb{R}^n)} \le C[u]_{H^s(\mathbb{R}^n)} \qquad q = \frac{2n}{n-2s}.$$

Poincaré inequality

$u \in H^s(\mathbb{R}^n)$ s.t. $u \equiv 0$ $\mathbb{R}^n \setminus \Omega$, Ω bounded.

$\|u\|_{L^2(\Omega)} \leq C [u]_{H^s(\mathbb{R}^n)}$.



Pf. 1. $\int_{\Omega^C} \frac{dy}{|x-y|^{n+2s}} \geq C$

($\int_{\Omega^C} > \int_{B_{2R} \setminus B_R}$ for $\Omega \subset B_R$

$|x-y| \leq 3R$).

$$|u(x)|^2 \leq \frac{1}{C} \int \frac{u(x)^2}{|x-y|^{n+2s}} dy \leq \frac{1}{C} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dy \quad \square$$

$W^{s,p}$ spaces. $p \in [1, +\infty)$, $W^{s,p}(\Omega)$.

$$W^{s,p}(\Omega) := \left\{ u \in L^p : \frac{u(x)-u(y)}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}$$

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p + \int_\Omega \int_\Omega \frac{(u(x)-u(y))^p}{|x-y|^{n+sp}} \right)^{1/p}.$$

$H^s(\Omega)$ and $\{u \in H^s(\mathbb{R}^n) \mid u \equiv 0 \ \mathbb{R}^n \setminus \Omega\}$ are different space.

Extension. $\Omega \subset \mathbb{R}^n$ open, $C^{0,1}$ with bonded bonday.

Then $W^{s,p}(\Omega)$ continuously embeded in $W^{s,p}(\mathbb{R}^n)$ i.e.

$\forall u \in W^{s,p}(\Omega)$. $\exists \hat{u} \in W^{s,p}(\mathbb{R}^n)$ st. $\hat{u}|_\Omega = u$ and.

$$\|\hat{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{s,p}(\Omega)}.$$

Open. Characterize the class of sets that are extension domains

Estimates.

Caccioppoli inequality

$(-\Delta)u = h$ let $\varphi = 1$ in $B_{r/2}$ $|\nabla\varphi| < Cr^{-1}$ $\varphi = 0$ B_r^c

$$\int (-\Delta u)\varphi^2 = \int h\varphi^2 \Rightarrow \int |\nabla u \cdot 2\varphi\nabla\varphi| \le \int_{B_r} h$$

$$\Rightarrow \int \underbrace{|\nabla u\varphi|^2}_{\int_{B_r}|\nabla u|^2} + \underbrace{|\varphi\nabla\varphi|^2}_{\text{estimate } \sim r^{-2}}.$$

$(-\Delta)^s u = h$ in B_r. $h \in L^1$,

Then
$$\int_{B_{9/10 r}}\int_{B_{9/10 r}} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} \le \sup_{B_r}|u|\,\|h\|_{L^1} + Cr^{n-2s}\underbrace{\|u\|_{L^\infty(\mathbb{R}^n)}}_{\sup_{B_r}|u|}$$

Pf. η cutoff. $\eta = 1$ in $B_{9r/10}$. $\eta = 0$ outside B_r, $|\nabla\eta| \le Cr^{-1}$

$$\iint_{\mathbb{R}^n\mathbb{R}^n} \frac{(u(x)-u(y))(\eta^2(x)u(x) - \eta^2(y)u(y))}{|x-y|^{n+2s}} = \int h\eta^2 u \le \sup_{B_r}|u|\,\|h\|_{L^1}$$

$$\iint_{\mathbb{R}^n\mathbb{R}^n} = \int_{B_r}\int_{B_r} + \underbrace{2\int_{B_r}\int_{\mathbb{R}^n\setminus B_r}}$$

$$(u(x)-u(y))(\eta^2(x)u(x) - \eta^2(y)u(y)) = (\eta(x)u(x) - \eta(y)u(y))^2 - u(x)u(y)(\eta(x)-\eta(y))^2$$

$$\int_{B_r}\int_{\mathbb{R}^n} \frac{u(x)u(y)(\eta(x)-\eta(y))^2}{|x-y|^{n+2s}} = \int_{B_r}\int_{|x-y|<r} \cdots + \int_{B_r}\int_{|x-y|>r}$$

$|x-y|<r$. $\qquad |\eta(x)-\eta(y)| \le C|x-y|$

$$\int_{B_r}\int_{|x-y|<r} \cdots \le C\|u\|_{L^\infty(\mathbb{R}^n)} \sup_{B_r}|u| \underbrace{\iint_{B_r, |x-y|<r} \frac{|x-y|^2}{|x-y|^{n+2s}}}$$

$$\int_0^\infty \frac{t^2}{t^{n+2s}} t^{n-1}\, dt < +\infty$$

$t^{1-2s} \qquad 1-2s > -1$.

$|x-y|>r$ pieces $\quad |y|<1$

$$\int_{B_r}\int_{|x-y|<r} \le C\|u\|_{L^1(\mathbb{R}^n)} r^{n-2s} \sup_{B_r}|u| . \qquad \square$$

Maximum Principle

$\Omega \subset \mathbb{R}^n$ bonded open. u supersolution $(-\Delta)^s u \ge 0$ w/ $u \ge 0$ in $\mathbb{R}^n \setminus \Omega$. Then $u \ge 0$ in $\mathbb{R}^n$.

Local maximum principle. (Kussi - Migione - Sine, 2013).

$$\begin{cases} (-\Delta)^s u = \mu & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \qquad \mu \in (W^{s,p}(\Omega))'.$$

$u \in W^{s,p}(\Omega)$

$$u(x_0) \le c\left(\int_0^r \frac{|\mu|(B_\rho(x_0))}{\rho^{n-sp}} \frac{1}{\rho} d\rho\right)^{\frac{1}{p}-1} + c\left(\fint_{B_r(x_0)} |u|^{q^*}\right)^{1/q^*}$$

Wolff potential $+ \left[r^{sp} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|u(x)|^{p-1}}{|x-x_0|^{n+sp}} \right]^{\frac{1}{p}-1}$.

Tail

Reference.

Di Nezza, Palatucci, Valdinoci hitchhiker.

Ros-Oton, Fernandez-Real book $\Leftarrow$

$$\int (u(x)-u(y))(v(x)-v(y))K(x-y)$$

$$K(x-y) \sim \frac{1}{|x-y|^{n+2s}} . \quad //$$